Chaos, Solitons and Fractals 116 (2018) 365–375

Contents lists available at ScienceDirect

Chaos, Solitons and Fractals

Nonlinear Science, and Nonequilibrium and Complex Phenomena

journal homepage: www.elsevier.com/locate/chaos

Riemann surface and Riemann theta function solutions of the discrete integrable hierarchy

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ARTICLE INFO

Article history:
Received 10 September 2017
Revised 4 February 2018
Accepted 10 September 2018

Keywords:
The discrete integrable equations
Riemann theta function solutions
Riemann surface

ABSTRACT

A hierarchy of discrete nonlinear evolution equations associated with a discrete 3×3 matrix spectral problem with two potentials is proposed by means of the Lenard recursion equations and zero-curvature equation. Based on the characteristic polynomial of Lax matrix for the hierarchy, we introduce a trigonal curve and study the properties of the corresponding three-sheeted Riemann surface, especially including arithmetic genus, holomorphic differentials. Base on the essential properties of the meromorphic functions ϕ_2, ϕ_3 and the Baker–Akhiezer function ψ_1, and their asymptotic behavior, we obtain Riemann theta function solutions of the entire discrete integrable hierarchy.

1. Introduction

Riemann theta function solutions describe the quasi-periodic behavior of nonlinear phenomenon or characteristic for the integrability of soliton equations. Moreover they can be used to find multi-soliton solutions, elliptic function solutions and others [1–3]. Over the past four decades, there have been fairly mature techniques to construct Riemann theta function solutions or quasi-periodic solutions of soliton equation associated with 2×2 matrix spectral problems, such as the KdV, the nonlinear Schrödinger, the sine-Gordon, the Toda equations and so on [4–9,28–34]. However, when our sight turns to the 3×3 spectral problems, their complexity considerably increases because of concerning the theory of trigonal curves [10–19] rather than the hyperelliptic cases for the 2×2 spectral problems. In [1,13–18,35,36], certain algebro-geometric solutions of the Boussinesq equation related to a third-order differential operator were found as special solutions of the Kadomtsev–Petviashvili equation or by the reduction theory of Riemann theta functions. In [20], Dickson et al. proposed an unified framework, which yields all quasi-periodic solutions of the entire Boussinesq hierarchy. In [21–24], Geng et al. further developed the method to deal with soliton equations associated with 3×3 matrix spectral problems, such as the modified Boussinesq, the Kaup–Kupershmidt, the coupled mKdV hierarchies, and others. The present paper can be viewed as the development of these approaches in the discrete case, which is important because it is quite scarce that the literature devoted to study Riemann theta function solutions of discrete soliton equations associated with 3×3 matrix spectral problems.

The main aim of the present paper is to derive the discrete integrable hierarchy associated with a 3×3 matrix spectral problem and to construct its Riemann theta function solutions with the aid of the theory of trigonal curves [10–12,19]. The first nontrivial member in the hierarchy is the discrete 2-potential system

$$\begin{aligned} u_{n,t} &= \quad u_{n+2}(1-u_{n+1}v_{n+1})(1-u_nv_n), \\ v_{n,t} &= \quad -v_{n-2}(1-u_{n-1}v_{n-1})(1-u_nv_n). \end{aligned} \tag{1.1}$$

This paper is organized as follows. In Section 2, we introduce a discrete 3×3 matrix spectral problem with two potentials and derive the discrete integrable hierarchy based on the Lenard recursion equations and zero-curvature equation. In Section 3, we introduce the Baker–Akhiezer function, the trigonal curve, and the corresponding three-sheeted Riemann surface with the help of the characteristic polynomial of Lax matrix for this hierarchy, from which the meromorphic functions on the Riemann surface are given. In Section 4, we study the essential properties of the meromorphic functions ϕ_2, ϕ_3 and the Baker–Akhiezer function ψ_1, and their asymptotic behavior. The last section is devoted to construct the Riemann theta function solutions of the discrete integrable hierarchy by employing three kinds of Abelian differentials.

2. The discrete integrable hierarchy

Throughout this paper we suppose the following hypothesis. Assume that u and v satisfies $uv \neq 0, u(\cdot,t),\ v(\cdot,t) \in \mathbb{C}^{\mathbb{Z}},\ t \in \mathbb{R},\ u(n,\cdot),\ v(n,\cdot) \in C^1(\mathbb{R}),\ n \in \mathbb{Z}$, where $\mathbb{C}^{\mathbb{Z}}$ denotes the set of all complex-valued sequences indexed by $\mathbb{Z}$.

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https://doi.org/10.1016/j.chaos.2018.09.014
0960-0779/

Let us define the shift operators E, E^{-1} and difference operator $\triangle$ by

$$Ef(n)=f(n+1),\quad E^{-1}f(n)=f(n-1),\quad \triangle f(n)=(E-1)f(n),$$

We usually write $f(n)=f,\ E^{\pm1}=E^{\pm},\ f(n\pm1)=f^{\pm}(n),\ f(n+k)=E^kf,\ k\in\mathbb{Z}.$ Consider the discrete 3×3 matrix spectral problem

$$E\psi=U\psi,\quad \psi=\begin{pmatrix}\psi_1\\ \psi_2\\ \psi_3\end{pmatrix},\quad U=\begin{pmatrix}0&\lambda&0\\ 1&0&u\\ v&0&1\end{pmatrix},\tag{2.1}$$

where u and v are two potentials, and λ is a constant spectral parameter. We first introduce the sequences $\hat{g}_j,\breve{g}_j$ recursively by

$$K\hat{g}_j=J\hat{g}_{j+1},\quad \hat{g}_j=(\hat{a}_j,\hat{b}_j,\hat{c}_j,\hat{d}_j,\hat{e}_j)^{\mathrm{T}},\quad j\geq0,\tag{2.2}$$

$$K\breve{g}_j=J\breve{g}_{j+1},\quad \breve{g}_j=(\breve{a}_j,\breve{b}_j,\breve{c}_j,\breve{d}_j,\breve{e}_j)^{\mathrm{T}},\quad j\geq0,\tag{2.3}$$

with the conditions $\hat{g}_j\mid_{(u,v)=0}=\breve{g}_j\mid_{(u,v)=0}=0,\ j\geq1,$ and starting points

$$\hat{g}_0=\begin{pmatrix}-uv^-\\ u\\ v^-\\ uv^{--}(u^-v^--1)\\ -1\end{pmatrix},\quad \breve{g}_0=\begin{pmatrix}1\\ 0\\ 0\\ -(E+1)^{-1}uv^-\\ 0\end{pmatrix},\tag{2.4}$$

where $(E+1)^{-1}(E+1)=(E+1)(E+1)^{-1}=1$, the initial conditions mean to identify constants of summation as zero, and the two difference operators K and J are defined as

$$K=\begin{pmatrix}EuE&E^2&0&uE^2&0\\ -E^{-1}v&0&-E^{-1}&-v&0\\ E^2-1&EvE&-u&0&0\\ 0&-v&uE&0&\triangle\\ vEuE-uE^{-1}v&vE^2&-uE^{-1}&E^2-1&0\end{pmatrix},$$

$$J=\begin{pmatrix}0&1&0&0&u\\ 0&0&-E&0&-vE\\ E^2-1&EvE&-u&0&0\\ 0&-v&uE&0&\triangle\\ vEuE-uE^{-1}v&vE^2&-uE^{-1}&E^2-1&0\end{pmatrix}.$$

Then $\hat{g}_j$ and $\breve{g}_j$ are uniquely determined by the recursion Eqs. (2.2) and (2.3), respectively, up to a term in $\ker J_n$, which is always assumed to be zero. For example,

$$\hat{g}_1=\begin{pmatrix}uv^-\hat{e}_1-(E^{-2}+1)u^{++}v^-(1-uv)(1-u^+v^+)\\ -u\hat{e}_1+u^{++}(1-u^+v^+)(1-uv)\\ -v^-\hat{e}_1+v^{---}(1-u^{--}v^{--})(1-u^-v^-)\\ \hat{d}_1\\ (E+1)uv^{--}(1-u^-v^-)\end{pmatrix},$$

$$\breve{g}_1=\begin{pmatrix}(E^2-1)^{-1}(u\breve{c}_1-v^+\breve{b}_1^{++})\\ u^+-u^2v-u(E+1)^{-1}u^{++}v^+\\ v^{--}-u(v^-)^2-v^-(E+1)^{-1}u^-v^{--}\\ uv^-\\ (E^2-1)^{-1}(u\breve{c}_1^--v\breve{b}_1^{++}+uv^-\breve{a}_1^--u^+v\breve{a}_1^{++})\end{pmatrix},$$

where

$$\begin{aligned}\hat{d}_1&=u^2(v^{--})^2(1-u^-v^-)^2-(1+E^{-2})u^{++}v^{--}\\ &\quad\times(1-u^-v^-)(1-uv)(1+u^+v^+)\\ &\quad+(1+E^{-1})u^+uv^-v^{--}(1-u^-v^-)(1-uv).\end{aligned}$$

In order to generate a hierarchy of nonlinear evolution equations associated with the spectral problem (2.1), we solve the stationary discrete zero-curvature equation

$$(EV)U-UV=0,\quad V=\begin{pmatrix}V_{11}&\lambda V_{12}&\lambda V_{13}\\ V_{21}&V_{22}&V_{23}\\ V_{31}&\lambda V_{32}&\lambda V_{33}\end{pmatrix},\tag{2.5}$$

which is equivalent to

$$n\in\mathbb{Z}.\quad \begin{aligned}&V_{12}^++vV_{13}^+-V_{21}=0,\\ &V_{11}^+-V_{22}=0,\\ &uV_{12}^++V_{13}^+-V_{23}=0,\\ &V_{22}^++vV_{23}^+-V_{11}-uV_{31}=0,\\ &V_{21}^+-V_{12}-uV_{32}=0,\\ &uV_{22}^++V_{23}^+-\lambda(V_{13}+uV_{33})=0,\\ &\lambda(V_{32}^++vV_{33}^+)-vV_{11}-V_{31}=0,\\ &V_{31}^+-vV_{12}-V_{32}=0,\\ &uV_{32}^++V_{33}^+-vV_{13}-V_{33}=0,\end{aligned}\tag{2.6}$$

where each entry $V_{ij}=V_{ij}(a,b,c,d,e)$ is a Laurent expansion in λ:

$$\begin{aligned}&V_{11}=d, && V_{12}=a, && V_{13}=b,\\ &V_{21}=a^++vb^+, && V_{22}=d^+, && V_{23}=ua^++b^+,\\ &V_{31}=v^-a^-+c^-, && V_{32}=c, && V_{33}=e,\end{aligned}\tag{2.7}$$

$$\begin{aligned}&a=\sum_{j\geq0}a_j\lambda^{-j},\quad b=\sum_{j\geq0}b_j\lambda^{-j},\quad c=\sum_{j\geq0}c_j\lambda^{-j}.\\ &d=\sum_{j\geq0}d_j\lambda^{-j},\quad e=\sum_{j\geq0}e_j\lambda^{-j}.\end{aligned}\tag{2.8}$$

A direct calculation shows that (2.6) and (2.7) imply the Lenard equation

$$KG=\lambda JG,\qquad G=(a,b,c,d,e)^{\mathrm{T}}.\tag{2.9}$$

Substituting (2.8) into (2.9) and collecting terms with the same powers of λ, we arrive at the Lenard recursion relation

$$KG_j=JG_{j+1},\qquad JG_0=0,\qquad j\geq0,\tag{2.10}$$

where $G_j=(a_j,b_j,c_j,d_j,e_j)^{\mathrm{T}}$. Since equation $JG_0=0$ has a solution

$$G_0=\alpha_0\hat{g}_0+\beta_0\breve{g}_0,\tag{2.11}$$

then G_j can be expressed as

$$G_j=\alpha_0\hat{g}_j+\beta_0\breve{g}_j+\alpha_1\hat{g}_{j-1}+\beta_1\breve{g}_{j-1}+\ldots+\alpha_j\hat{g}_0+\beta_j\breve{g}_0,\quad j\geq0,\tag{2.12}$$

where α_j and β_j are arbitrary constants.

Let ψ satisfy the discrete spectral problem (2.1) and an auxiliary problem

$$\psi_{t_r}=\widetilde{V}^{(r)}\psi,\qquad \widetilde{V}^{(r)}=\begin{pmatrix}\widetilde{V}_{11}^{(r)}&\lambda\widetilde{V}_{12}^{(r)}&\lambda\widetilde{V}_{13}^{(r)}\\ \widetilde{V}_{21}^{(r)}&\widetilde{V}_{22}^{(r)}&\widetilde{V}_{23}^{(r)}\\ \widetilde{V}_{31}^{(r)}&\lambda\widetilde{V}_{32}^{(r)}&\lambda\widetilde{V}_{33}^{(r)}\end{pmatrix},\tag{2.13}$$

where each entry $\widetilde{V}_{ij}^{(r)}=V_{ij}(\tilde{a}^{(r)},\tilde{b}^{(r)},\tilde{c}^{(r)},\tilde{d}^{(r)},\tilde{e}^{(r)})$,

$$\begin{aligned}&\tilde{a}^{(r)}=\sum_{j=0}^{r}\tilde{a}_j\lambda^{r-j},\quad \tilde{b}^{(r)}=\sum_{j=0}^{r}\tilde{b}_j\lambda^{r-j},\quad \tilde{c}^{(r)}=\sum_{j=0}^{r}\tilde{c}_j\lambda^{r-j},\\ &\tilde{d}^{(r)}=\sum_{j=0}^{r}\tilde{d}_j\lambda^{r-j},\quad \tilde{e}^{(r)}=\sum_{j=0}^{r}\tilde{e}_j\lambda^{r-j},\end{aligned}\tag{2.14}$$

with $\widetilde{G}_j=(\tilde{a}_j,\tilde{b}_j,\tilde{c}_j,\tilde{d}_j,\tilde{e}_j)^{\mathrm{T}}$ determined by

$$\widetilde{G}_j=\tilde{\alpha}_0\hat{g}_j+\tilde{\alpha}_1\hat{g}_{j-1}+\ldots+\tilde{\alpha}_j\hat{g}_0,\quad j\geq0,\tag{2.15}$$

and the constants $\{\tilde{\alpha}_j\}$ and $\{\alpha_j\}$ are independent of each other. Then the compatibility condition of (2.1) and (2.13) yields the discrete zero-curvature equation, $U_{t_r}=(E\widetilde{V}^{(r)})U-U\widetilde{V}^{(r)}$, which is equivalent to the discrete integrable hierarchy

$$(u_{t_r},v_{t_r})^{\mathrm{T}}=X_r,\quad r\geq0,\tag{2.16}$$

with the vector fields

$$X_j=\mathcal{P}\big(K_n\widetilde{G}_j\big)=\mathcal{P}\big(J_n\widetilde{G}_{j+1}\big),\quad j\geq0,\tag{2.17}$$