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Malliavin calculus for subordinated Lévy process

Hi Jun Choe, Ji Min Lee*, Jung-Kyung Lee

Department of Mathematics, Yonsei University, 50 Yonsei-ro, Seodaemun-gu, Seoul, 03722 South Korea

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ABSTRACT

We develop a chaos expansion for a subordinated Lévy process. This expansion is expressed in terms of Itô's multiple integral expansion. Considering the jumps occurring due to an underlying process and a subordinator, a mixed chaotic representation is proposed. This representation provides the definition of the Malliavin derivative, which is characterized by increment quotients. Moreover, we introduce a new Clark–Ocone expansion formula for the subordinated Lévy process and provide applications for risk-free hedging in a designed model.

1. Introduction

The Lévy process, introduced by Paul Lévy, played a central role in the study of stochastic processes in early 1900s, because the Lévy process constitutes a broad class of probabilistic processes that can be continuous, sometimes discontinuous, and purely discontinuous. This extremely robust class of processes exhibits many interesting phenomena in the theories of stochastic and potential analysis. In physics, Lévy processes are used to study turbulence, laser cooling, and quantum field theory. Engineers use them in network, queue, and dam research. In mathematical finance, Lévy processes are becoming extremely useful because they can be used to explain the observed reality of financial markets in a more accurate way than the Brownian motion based model. Many practitioners have used Lévy processes for modeling asset prices because asset price processes have jumps or spikes in the "real world" environment. A comprehensive overview of various applications of Lévy processes can be found in studies conducted by Kyprianou [10], Prabhu [20], and Barndorff-Nielsen et al. [2].

In fact, Lévy processes are a family of stochastic processes, and they have many types: linear Brownian motions, Poisson processes, compound Poisson processes, subordinators, stable processes, Gamma processes, and Inverse Gaussian processes.

This paper focuses on subordinators, which transform a stochastic process into a new stochastic process. Subordination is applicable to Lévy processes that are used as a stochastic time change, which, by itself, is an almost surely increasing Lévy process. The new process is considered subordinate to the original one, and the economic interpretation of the time change is based on the idea that "business time" (subordinator) may run faster than "calendar time" (physical time) during some periods. Hence, subordinators can only be an approximation of the real behavior of asset prices. The subordination of Lévy processes has many important applications. In mathematical finance, it is a time change that models the flow of information and measures the trade volume time as opposed to real time. There exists a study wherein a subordinated Lévy process was applied to commodities [9].

We emphasize that the subordinated Lévy process plays a crucial role in fluctuation theory of general Lévy processes as the renewal processes are a key in random-walk theory. From the moment when stock trading occurs, we observe a sharp rise and decline in asset prices. In fact, the graph of the stock price appears to show a continuous probability process, but it can be also be perceived as a series of countless jumps that represent buying and selling. In a very short moment of trading, one tick (the smallest price fluctuation unit) shows that the fluctuating index represents rises of the buy-order (bid price) and downs of the sell-order (ask price), respectively. Therefore, the bid-ask price at each moment of the transaction can be explained in terms of the stock trading process. Thus, subordinated Lévy processes are more likely to explain the minimum price fluctuations and uncertainties in a market economy during the financial transaction process.

Malliavin calculus was first introduced by Paul Malliavin [14] as an infinite dimensional integration by parts that is adapted for

* Corresponding author.
E-mail addresses: choe@yonsei.ac.kr (H.J. Choe), bluegee@naver.com, jmlee524@yonsei.ac.kr (J.M. Lee), lee.jk@yonsei.ac.kr (J.-K. Lee).

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Lévy processes by Nunno et al. [7], Nualart [16] and Solé [22], mainly because it is used to prove the smoothness of stochastic differential equations for solutions' density.

In probability theory and related fields, Malliavin calculus is a set of mathematical techniques and notions; this set extends the field of calculus of variations from deterministic functions to stochastic processes. In particular, the computation of random variable derivatives becomes possible. The calculus allows for the integration by parts with random variables, and this operation is used in mathematical finance to compute the sensitivities of financial derivatives. Malliavin calculus in Brownian motion realizes this in two ways that are equivalent: one as a weak derivative in canonical space and the other through Wiener chaos. However, Lévy processes generally do not have the chaotic representation property used in the Brownian motions, Poisson processes, or so-called normal martingales [15]. Previous work by Nualart and Schoutens [17], which included a type of chaotic representation property for Lévy processes, has enabled us to define a Malliavin derivative using the chaotic approach. It is necessary to introduce a two-parameter random measure associated with the Lévy process, and the representation of a functional derivative is formulated using multiple integrals with respect to this random measure. Additionally, we consider the multiple integral chaotic representation of a Lévy process initially suggested by Itô [8] and the symmetric L^2 base Martingale representation under the spirit of Itô [8] considered by Løkka [13].

The Malliavin derivative of a random variable f in L^2 is defined by expanding it first in terms of Gaussian random variables that are parametrized by Hilbert space elements and by the expansions of formal differentiation. The Skorohod integral is an adjoint operation to this Malliavin derivative. The stochastic calculus techniques of variations on the Wiener space enabled the development a stochastic calculus for the Skorohod integral [18], which extends the classical Itô calculus introduced in the 1940s, most properties of the Itô stochastic integral. In addition, the Clark–Ocone formula is an explicit stochastic integral representation for random variables in terms of Malliavin derivatives that are crucial when applied to mathematical finance studied by David León [12]. It is worth mentioning that most aforementioned studies dealt with pure jump Lévy processes or combinations of the Brownian motion and Poisson processes. The general Lévy processes, which satisfy certain conditions, were considered for a normal martingale case [1] and [15]. In fact, Solé et al. [22] studied the characteristics of canonical Malliavin derivatives suggested by Løkka [13] in terms of a Poisson random measures. Following the argument of Solé et al. [22], we also characterize Malliavin derivatives using the increment quotient.

The main contributions of this paper are the derivation of the chaos representation property, Malliavin derivative, Skorohod integral, and Clark-Ocone formula of subordinated Lévy processes. Moreover, we provide applications for risk-free hedging using a derived Clark–Ocone formula in a subordinated Lévy process.

We introduce refined definitions for the Malliavin derivatives of a subordinated Lévy process with respect to the Brownian motion, pure jumps of the original process, and pure jumps of the subordinator. We conventionally obtain the integration by parts formula and the Skorohod integral with the chaos expansions for the subordinated Lévy process. We derive the Clark-Ocone formula, which also follows from the definition of Malliavin derivatives and approximations of the subordinated Lévy process. Unfortunately, the subordinated Lévy processes correspond to incomplete models. A perfect hedge cannot be obtained, and there is always a residual risk that cannot be hedged; however, but risk-free hedging is possible when a derived Clark–Ocone formula is used in a subordinated Lévy process.

The remainder of this paper is organized as follows. In Section 2, we introduce subordination, the Lévy–Khintchine formula, and relevant mathematical preliminaries. In Section 3, representations are derived by multiple integrals that reflect jumps in the underlying processes and subordinators while also proving their orthogonal decomposition of the L^2 space. In Section 4, we calculate new types of Malliavin derivatives, while in Section 5, we derive the Clark–Ocone formula for the subordinated Lévy process. Finally, in Section 6, we calculate risk-free hedging to an application available in finance.

2. Subordination of Lévy process

This section reviews several basic concepts related to Lévy process and subordination (refer to Bertoin [3] and Sato [21]).

A stochastic process $\{X_t: t \geq 0\}$ on $\mathbb{R}$ is said to be a Lévy Process in a complete probability space $(\Omega, \mathcal{F}, \mathbf{P})$ if it satisfies the following properties:

• **Independence of increments**: For any $0 \leq t_1 < t_2 < \ldots < \infty$, random variables $X_{t_1} - X_{t_0}, X_{t_2} - X_{t_1}, \cdots, X_{t_n} - X_{t_{n-1}}$ are independent.

• **Stationary increments**: The distribution of $X_{s+t} - X_t$ dose not depend on s.

• **Continuity in probability**: For any $\epsilon > 0$ and $t \geq 0$ it holds that $\lim_{s\to 0} P(|X_{t+s} - X_t| > \epsilon) = 0$. If X_t is a Lévy process then $t \mapsto X_t$ is Càdlàc that almost surely right-continuous with left limits.

Let X_t be a Lévy process and denote by

$$X_{t-} = \lim_{s\to t, s<t} X_s, \quad t > 0,$$

the left limit process and by $\Delta X_t = X_t - X_{t-}$ the jump size at time t.

The distribution of a Lévy process is characterized by its characteristic function of all infinitely divisible distributions, which is given by the Lévy–Khintchine formula. If $\{X_t: t \geq 0\}$ is a Lévy process, then its characteristic function of the form

$$\mathbb{E}[e^{i\xi X_t}] = exp(t\Psi_1(\xi)),$$

where $e^{\Psi_1(\xi)}$ is the characteristic function of X_1. The function $\Psi_1(\xi) = log(\widehat{\mu}(\xi))$ is called the characteristic exponent. If we set μ^t for probability measure of the Lévy process X_t and $\mu = \mu^1$, then the Lévy–Khintchine formula of characteristic function holds:

$$\hat{\mu}(\xi) = \exp\left(-\frac{1}{2}A_1|\xi|^2 + ib_1\xi + \int_{\mathbb{R}}(e^{ix\xi} - 1 - ix\xi\mathbf{1}_{|x|<1})\,\nu_1(dx)\right), \quad \xi \in \mathbb{R}, \tag{2.1}$$

where $A_1 \geq 0$ is a Gaussian variance, $b_1 \in \mathbb{R}$ is a drift term, $\mathbf{1}_B(x)$ is the indicator function of a set B, and $\nu_1 \in \mathbb{R}$ is a measure on $\mathbb{R}\backslash\{0\}$ with $\int_{\mathbb{R}}(1 \wedge x^2)\nu(dx) < \infty$. The measure ν_1 is called the Lévy measure of X_t. The Lévy process comprises three independent components : a Brownian motion, a drift term, and a superposition of independent Poisson processes with different jump sizes. These three components and the Lévy - Khintchine formula are determined by the Lévy–Khintchine triplet (A_1, b_1, ν_1).

The process X_t admits a Lévy-Itô decomposition

$$X_t = b_1 t + \sqrt{A_1}W_t + \int_{(0,t]\times\{|x|>1\}} x dN(s,x) + \lim_{\epsilon\to 0}\int_{(0,t]\times\{\epsilon<|x|\leq 1\}} x d\widetilde{N}(s,x), \tag{2.2}$$

where $\{W_t: t \geq 0\}$ is the standard Brownian motion. We let $\mathfrak{B}((0,\infty)\times\mathbb{R}\setminus\{0\})$ be a Borel σ-algebra of $(0, \infty)\times\mathbb{R}\setminus\{0\}$.

$$N(B) = \sharp\{t : (t, \Delta X_t) \in B\}, \quad B \in \mathfrak{B}((0,\infty)\times\mathbb{R}\setminus\{0\}), \tag{2.3}$$

is the jump measure of the process, where $\Delta X_t = X_t - X_{t-}$, $\sharp A$ denotes the cardinal of a set A, and the compensated Poisson random measure is

$$d\tilde{N}(t,x) = dN(t,x) - dt d\nu(x). \tag{2.4}$$