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Tuned communicability metrics in networks. The case of alternative routes for urban traffic

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ABSTRACT

We generalize here the communicability metric on graphs/networks to include a tuning parameter that accounts for the level of edge "deterioration". This generalized metric covers a wide range of realistic scenarios in networks, which includes shortest-path metric as a particular case. We study the communicability metric on an urban street network, and show that communicability shortest paths in this city accounts for most of the traffic between series of origin-destination points. Particularly, we show that the traffic flow and congestion in the shortest communicability paths is much bigger than in the corresponding shortest paths. This indicates that under certain conditions drivers in a city avoid long paths but also avoid the most interconnected street intersections, which typically may be the most congested ones. We develop here a diffusion-like model on the network based on a particle-hopping scheme inspired by "tight-binding" quantum mechanical Hamiltonian, which offers a solid explanation on why traffic is diverted through the shortest communicability routes instead of the shortest-paths.

1. Introduction

The study of complex networks, which represent the topological framework of complex systems, has found a wide variety of applications in sciences, engineering and the analysis of social systems [1,2]. From a mathematical point of view a complex network is a graph with nontrivial structure, typically having a large number of nodes and being sparse [1–3]. It can be argued that the main reason for the existence of such networked structures in complex systems is to transmit "items" from one part of the system to another [4]. Such "items" may refer to electrons (molecular networks), water (brain networks), gossip (social networks), mass/energy (ecological networks), or any other thing that complex systems need for their functioning [1]. In such context of trading items among nodes of a complex system it is vital to understand the metric properties of the underlying system, i.e., the network. By far, the most studied metric in graphs [5] and networks [1–4] is the so-called *shortest path distance.* The shortest path between two nodes in a network is the shortest–in terms of the number of edges–among all the sequences of different nodes and edges connecting the origin and the destination. The shortest path in an undirected graph is a proper metric in the sense that it fulfills the following axioms [6]: (i) $d(x, y) \geq 0$ (*nonnegativity*); (ii) $d(x, y) = 0$ if and only if $x = y$ (*identity of indiscernibles*); (iii) $d(x, y) = d(y, x)$ (*symmetry*); (iv) $d(x, y) \leq d(x, z) + d(z, y)$ (*triangle inequality*).

Another metric in graphs is the so-called *resistance distance* [7–9], which is defined on the basis of the Moore–Penrose pseudo-inverse of the Laplacian matrix. That is, the resistance distance between the nodes i and j in a graph is defined as: $\Omega_{ij} = L^{\dagger}_{ii} + L^{\dagger}_{jj} - 2L^{\dagger}_{ij}$, where $L^{\dagger}_{ii}$ is the diagonal entry of the pseudo-inverse of the Laplacian matrix corresponding to the node i. The Laplacian matrix is defined as the difference between a diagonal matrix of node degree K and the adjacency matrix of the graph A, i.e., $L = K - A$. The term resistance distance comes from the fact that if we place a fixed electrical resistor on each edge of a network and we connect a battery across the nodes, then the effective resistance between them obtained by using the Kirchhoff and Ohm laws is given by Ω_{ij}. In graph-theoretic terms the resistance distance is a weighted sum of all paths between the origin and destination nodes, which means that it is identical to the shortest path metric when the graph is acyclic, i.e., in trees. The resistance distance is related to the *commuting time* $C(i, j)$ of a random walker between the nodes i and j on the graph by: $C(i, j) = 2m\Omega_{ij}$, where m is the number of edges in the graph [10–12]. Thus, the commuting time is also a proper metric in the graph [6].

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Both shortest path and resistance distance are *intrinsic metrics* of graphs and networks. That is, they "emerge" from the proper structure of the graphs without imposing any particular embedding to them. In another category we can find those metrics which are produced by an imposed embedding of the graph into a certain geometry. The most studied of these *extrinsic metrics* is the one arising from the embedding of a network in a hyperbolic space [13]. In this work we deal only with intrinsic (natural) metrics of graphs. In this category of intrinsic metrics we can also find *Chebotarev–Shamis metric* [14], which is defined as: $d_{ij}^{\alpha} = \frac{1}{2}\left(q_{ii}^{\alpha} + q_{jj}^{\alpha} - 2q_{ij}^{\alpha}\right)$, where q_{ii}^{α} is the corresponding entry of the matrix $Q = (I + \alpha L)^{-1}$. As proved by Chebotarev and Shamis their metric is the resistance metric of a given weighted multigraph [14,15]. Indeed, this metric becomes the resistance metric as $\alpha \to \infty$. A different metric is the so-called *communicability distance* [16,17], which is defined on the basis of the communicability function of a graph [18–20]. It is defined as: $\xi_{ij}^2 = G_{ii} + G_{jj} - 2G_{ij}$, where the communicability term G_{ij} is the corresponding entry of the matrix $G = \exp(A)$. This metric accounts for the "quality" of the communication routes between two nodes in a network, where the self-communicability terms G_{ii} and G_{jj} account for the number of routes in which items can get lost by returning to its originator [21] and G_{ij} accounts for the routes that connect the origin with the destination [18]. Here by routes we mean a walk of a given length as explained in the next section of this work. This metric induces an embedding of a graph in a Euclidean n-sphere [22,23]. That is, the communicability distance matrix is circum-Euclidean.

Although there is great popularity of the shortest path metric in the study of graphs and metrics, there is neither theoretical nor empirical support that it is the preferred way in which items are delivered in networks. In particular, in cases where the global topology of the network is not known, it is difficult to believe that the items can navigate in a way which mainly use the shortest path. In addition, in many real-world complex networks, most of the time, items travel across the network in a diffusion-like way, which does not involve the shortest path as the main route of delivery. Thinking about random walks, and consequently in terms of resistance distances, is an alternative method which has more foundations in the basis of the diffusive nature of network processes. However, as von Luxburg [24] has proved for large graphs, the commute distance converges to an expression that does not take into account the structure of the graph at all. As a consequence, it is "*completely meaningless as a distance function on the graph*" [24], which dramatically limits the use of resistance-like distances for large graphs, which are the type most frequently studied in network theory. Consequently, we focus our attention here in the communicability metric, which we generalize to include a parameter that allows us to recover the shortest path metric as a particular case.

Our main goal in this work is to generalize the communicability metric to include a tuning parameter that allows us to model different realistic scenarios for the flow of items on networks. In particular, we prove analytically that metrics of this type can be obtained from a few different matrix functions of the adjacency matrix of a network. However, the communicability metric based on the exponential of the adjacency matrix displays a large number of advantages over the others. In particular, it can cover a wider range of realistic scenarios in networks, while the other metrics are very close to the shortest path. Finally, we consider the communicability metric on both random spatial networks and an urban street network. In the last case, we show that communicability shortest paths in this city account for most of the traffic between sets of origin-destination points. Particularly, we show that for certain values of the tuning parameter, the traffic flow and congestion in the shortest communicability paths is much bigger than in the corresponding shortest path. This indicates that under certain conditions–analyzed in this paper–drivers in a city travel through paths which compromise of keeping the path short, however avoiding the most interconnected street intersections, which typically may be the most congested ones. We show here that these paths are well described by a diffusion-like process on the network based on a particle-hopping scheme, which is developed here for the first time.

2. Preliminaries

Here we present the definitions, notations, and properties associated with graphs to make this work self-contained. We consider only simple, undirected graphs $\Gamma = (V, E)$ with n nodes (vertices) and m edges. The notation used in the paper is the standard in graph theory and the reader is referred to the monograph [1] for details. An important concept to be used across this paper is the one of walks. A *walk* of length k in Γ is a set of nodes $i_1, i_2, \ldots, i_k, i_{k+1}$ such that for all $1 \le l \le k$, $(i_l, i_{l+1}) \in E$. A *closed walk* is a walk for which $i_1 = i_{k+1}$. A *path* is a walk with no repeated nodes/edges. Among all the paths connecting two nodes, the one with the shortest length is known as the *shortest path* between the two nodes. The length of the shortest path is named here as the *(topological) shortest path distance* between the corresponding nodes. A graph is *connected* if there is a path connecting every pair of nodes.

Let A be the adjacency matrix of the graph Γ. For simple graphs A is symmetric and thus its eigenvalues are real, which we label here in non-increasing order: $\lambda_1 \ge \lambda_2 \ge \ldots \ge \lambda_n$. We will consider the spectral decomposition of A: $= V\Lambda V^T$, where Λ is a diagonal matrix containing the eigenvalues of A and $V = [\vec{\psi}_1, \ldots, \vec{\psi}_n]$ is orthogonal, where $\vec{\psi}_i$ is an eigenvector associated with λ_i. As the graphs considered here are connected, A is irreducible and from the Perron–Frobenius theorem we can deduce that $\lambda_1 > \lambda_2$ and that the leading eigenvector $\vec{\psi}_1$, can be chosen such that its components $\psi_1(p)$ are positive for all $p \in V$. It is known that $(A^k)_{pq}$ counts the number of walks of length k between p and q.

2.1. Communicability function and external stress on networks

An important quantity for studying communication processes in networks has been defined as the communicability function [18]. It is defined as follows. Let p and q be two nodes of Γ. The communicability function between these two nodes is defined as

$$G_{pq} = \sum_{k=0}^{\infty} \frac{\left(A^k\right)_{pq}}{k!} = (\exp(A))_{pq} = \sum_{j=1}^{n} e^{\lambda_j} \psi_j(p) \psi_j(q).$$

It counts the total number of walks starting at node p and ending at node q, weighted in decreasing order of their length by a factor of $\frac{1}{k!}$; therefore it is considering shorter walks as more influential than longer ones. The matrix exponential is an example of a general class of matrix functions. In general we can consider any matrix function that can be expressed as a weighted sum of all the different powers of the adjacency matrix of the graph. That is,

$$(f(A))_{pq} = \sum_{k=0}^{\infty} c_k \left(A^k\right)_{pq}, \tag{2.1}$$

where c_k are coefficients giving more weight to the shorter than to the longer walks, and making the series converge. Some examples of these functions are given in Table 1.

Remark 1. Functions like $\cosh(A)$, $\sinh(A)$, $\cos(A)$, $\sin(A)$ are not considered as "complete" communicability functions as they do not account for all walks in the graph, i.e., they account only for even-