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Dendrite-type attractors of IFSs formed by two injective functions

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ABSTRACT

The aim of this paper is to study the dendrite-type attractors of an iterated function system formed by two injective functions. We consider (X, d) a complete metric space and $\mathcal{S} = (X, \{f_0, f_1\})$ an iterated function system (IFS), where $f_0, f_1 : X \longrightarrow X$ are injective functions and A is the attractor of $\mathcal{S}$. Moreover, we suppose that $f_0(A) \cap f_1(A) = \{a\}$ and $\{a\} = \pi(0^m 1^\infty) = \pi(1^n 0^\infty)$ with $m, n \geq 1$, where π is the canonical projection on the attractor. We compute the connected components of the sets $A \backslash \{\pi(0^\infty)\}$, $A \backslash \{\pi(1^\infty)\}$, $A \backslash \{\pi(0^m 1^\infty) = \pi(1^n 0^\infty)\}$ and deduce there are infinitely-many (countably) non-homeomorphic dendrite-type attractors of iterated function systems formed by two injective functions.

1. Introduction

Iterated function systems were conceived in the present form by J. Hutchinson in [9], popularized by M. Barnsley in [1] and are one of the most common and general ways to generate fractals. Many of the important examples of functions and sets with special and unusual properties turn out to be fractal sets or functions whose graphs are fractal sets and a great part of them are attractors of iterated function systems. There is a current effort to extend the classical Hutchinson's framework to more general spaces and infinite iterated function systems or, more generally, to multifunction systems and to study them ([10,11,14,15,17]). Such example can be found in [12], where the Lipscomb's space, which is an important example in dimension theory, can be obtained as an attractor of an infinite iterated function system defined in a very general setting. In those settings the attractor can be a closed bounded set, in contrast with the classical theory where only compact sets are considered. Although the fractal sets are defined with measure theory, being sets with non-integer Hausdorff dimension ([6–8,18,22]), it turns out that they have interesting topological properties ([3–5]). The topological properties of fractal sets have a great importance in analysis on fractals as we can see in ([5,10,11]). Generalized iterated function systems can be found in ([13,18–20]). Topological versions of an iterated function system have been studied in ([2,16,21]).

In this article we intend to characterize the dendrites which are attractors of iterated function systems composed by two injective functions. For a metric space (X, d), we denote by $\mathcal{K}^*(X)$ the set of nonempty compact subsets of X.

Definition 1.1. Let (X, d) be a metric space. The function $h : \mathcal{K}^*(X) \times \mathcal{K}^*(X) \longrightarrow [0, +\infty)$ defined by $h(A, B) = \max(d(A, B), d(B, A))$, where $d(A, B) = \sup_{x \in A} d(x, B) = \sup_{x \in A}\left(\inf_{y \in B} d(x, y)\right)$ is called the *Hausdorff-Pompeiu metric*.

Remark 1.1 ([18]). The space $(\mathcal{K}^*(X), h)$ is complete if (X, d) is complete, compact if (X, d) is compact and separable if (X, d) is separable.

Definition 1.2. Let (X, d) be a metric space. For a function $f : X \longrightarrow X$ let us denote by $Lip(f) \in [0, +\infty]$ the *Lipschitz constant* associated to f, which is $Lip(f) = \sup_{x,y \in X\ ;\ x \neq y} \frac{d(f(x), f(y))}{d(x, y)}$. We say that f is a *Lipschitz function* if $Lip(f) < +\infty$ and a *contraction* if $Lip(f) < 1$.

Definition 1.3. An *iterated function system (IFS)* on a complete metric space (X, d) consists of a finite family of contractions $\{f_k\}_{k=\overline{1,n}}$, $f_k : X \longrightarrow X$ for every $k \in \{1, 2, \ldots, n\}$ and it is denoted by $\mathcal{S} = (X, \{f_k\}_{k=\overline{1,n}})$.

Definition 1.4. For an iterated function system $\mathcal{S} = (X, \{f_k\}_{k=\overline{1,n}})$, the function $F_{\mathcal{S}} : \mathcal{K}^*(X) \longrightarrow \mathcal{K}^*(X)$ defined by $F_{\mathcal{S}}(B) = \bigcup_{k=1}^{n} f_k(B)$ is called the *fractal operator* associated to the iterated function system $\mathcal{S}$.

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Remark 1.2 ([1]). The function $F_{\mathcal{S}}$ is a contraction satisfying $Lip(F_{\mathcal{S}}) \leq \max_{k=\overline{1,n}} Lip(f_k)$.

Using Banach's contraction theorem there exists, for an iterated function system $\mathcal{S} = (X, \{f_k\}_{k=\overline{1,n}})$, a unique set $A \in \mathcal{K}^*(X)$ such that $F_{\mathcal{S}}(A) = A$, which is called the *attractor* of the iterated function system $\mathcal{S}$. More precisely we have the following well-known result.

Theorem 1.1 ([1,8,18]). *Let (X, d) be a complete metric space and $\mathcal{S} = (X, \{f_k\}_{k=\overline{1,n}})$ an iterated function system with $c = \max_{k=\overline{1,n}} Lip(f_k) < 1$. Then there exists a unique set $A = A(\mathcal{S}) \in \mathcal{K}^*(X)$ such that $F_{\mathcal{S}}(A) = A$. Moreover, for any $H_0 \in \mathcal{K}^*(X)$ the sequence $(H_n)_{n\geq 1}$ defined by $H_{n+1} = F_{\mathcal{S}}(H_n)$ is convergent to A. For the speed of the convergence we have the following estimate: $h(H_n, A) \leq \frac{c^n}{1-c} h(H_0, H_1)$ for every $n \geq 1$.*

Now we briefly present the shift space of an iterated function system. For more details one can see [11]. We start with some set notations: $\mathbb{N}$ denotes the set of natural numbers, $\mathbb{N}^* = \mathbb{N} \setminus \{0\}$, $\mathbb{N}_n^* = \{1, 2, \ldots, n\}$. For two nonempty sets A and B, B^A denotes the set of functions from A to B. By $\Lambda = \Lambda(B)$ we will understand the set $B^{\mathbb{N}^*}$ and by $\Lambda_n = \Lambda_n(B)$ we will understand the set $B^{\mathbb{N}_n^*}$. The elements of $\Lambda = \Lambda(B) = B^{\mathbb{N}^*}$ will be written as infinite words $\omega = \omega_1\omega_2 \ldots \omega_m\omega_{m+1} \ldots$, where $\omega_m \in B$ and the elements of $\Lambda_n = \Lambda_n(B) = B^{\mathbb{N}_n^*}$ will be written as finite words $\omega = \omega_1\omega_2 \ldots \omega_n$. By λ we will understand the empty word. Let us remark that $\Lambda_0(B) = \{\lambda\}$. By $\Lambda^* = \Lambda^*(B)$ we will understand the set of all finite words $\Lambda^* = \Lambda^*(B) = \bigcup_{n\geq 0} \Lambda_n(B)$.

We denote by $|\omega|$ the length of the word ω. An element of $\Lambda = \Lambda(B)$ is said to have length $+\infty$. If $\omega = \omega_1\omega_2 \ldots \omega_m\omega_{m+1} \ldots$ or if $\omega = \omega_1\omega_2 \ldots \omega_n$ and $n \geq m$, then $[\omega]_m := \omega_1\omega_2 \ldots \omega_m$. More generally, if $l < m$ then $[\omega]_m^l = \omega_{l+1}\omega_{l+2} \ldots \omega_m$ and we have $[\omega]_m = [\omega]_l[\omega]_m^l$ for every $\omega \in \Lambda_n(B)$, where $n \geq m > l \geq 1$. For two words $\alpha, \beta \in \Lambda^*(B) \cup \Lambda(B)$, $\alpha \prec \beta$ means that $|\alpha| \leq |\beta|$ and $[\beta]_{|\alpha|} = \alpha$. For $\alpha \in \Lambda_n(B)$ and $\beta \in \Lambda_m(B)$ or $\beta \in \Lambda(B)$ by $\alpha\beta$ we will understand the joining of the words α and β, namely $\alpha\beta = \alpha_1\alpha_2 \ldots \alpha_n\beta_1\beta_2 \ldots \beta_m$ and respectively $\alpha\beta = \alpha_1\alpha_2 \ldots \alpha_n\beta_1\beta_2 \ldots \beta_m\beta_{m+1} \ldots$.

On $\Lambda = \Lambda(\mathbb{N}_n^*) = (\mathbb{N}_n^*)^{\mathbb{N}^*}$, we can consider the metric $d_s(\alpha, \beta) = \sum_{k=1}^{\infty} \frac{1-\delta_{\alpha_k}^{\beta_k}}{3^k}$, where $\delta_x^y = \begin{cases} 1, \text{ if } x = y \\ 0, \text{ if } x \neq y \end{cases}$ and $\alpha = \alpha_1\alpha_2 \ldots$, $\beta = \beta_1\beta_2 \ldots$.

Let (X, d) be a complete metric space, $\mathcal{S} = (X, \{f_k\}_{k=\overline{1,n}})$ an iterated function system on X and $A = A(\mathcal{S})$ the attractor of the iterated function system $\mathcal{S}$. For $\omega = \omega_1\omega_2 \ldots \omega_m \in \Lambda_m(\mathbb{N}_n^*)$, f_ω denotes $f_{\omega_1} \circ f_{\omega_2} \circ \ldots \circ f_{\omega_m}$, where f_{ω_m} is applied first and H_ω denotes $f_\omega(H)$ for a subset $H \subset X$. By H_λ we will understand the set H. In particular $A_\omega = f_\omega(A)$. Moreover, we denote by $0^\infty = 000\ldots \in \Lambda(\{0, 1\})$ and $1^\infty = 111\ldots \in \Lambda(\{0, 1\})$.

The main results concerning the relation between the attractor of an iterated function system and the shift space is contained in the following theorem. The function $\pi : \Lambda \longrightarrow A = A(\mathcal{S})$ from the theorem below is called the *canonical projection* from the shift space onto the attractor of an iterated function system $\mathcal{S}$.

Theorem 1.2 ([11]). *Let (X, d) be a complete metric space. If $A = A(\mathcal{S})$ is the attractor of an iterated function system $\mathcal{S} = (X, \{f_k\}_{k=\overline{1,n}})$ with $c = \max_{k=\overline{1,n}} Lip(f_k) < 1$, then:*

(1) For every $\omega \in \Lambda = \Lambda(\mathbb{N}_n^*)$ we have $A_{[\omega]_{m+1}} \subset A_{[\omega]_m}$ and $d(A_{[\omega]_m}) \longrightarrow 0$ when $m \longrightarrow \infty$. More precisely $d(A_{[\omega]_m}) \leq c^m d(A)$, where $d(M) = \sup_{x,y\in M} d(x, y)$ is the diameter of a set M.

(2) If a_ω is defined by $\{a_\omega\} = \bigcap_{m\geq 1} A_{[\omega]_m}$, then $d(e_{[\omega]_m}, a_\omega) \longrightarrow 0$ when $m \longrightarrow \infty$, where $e_{[\omega]_m}$ is the unique fixed point of $f_{[\omega]_m}$.

(3) $A = \bigcup_{\omega\in\Lambda} \{a_\omega\}$, $A_\alpha = \bigcup_{\omega\in\Lambda} \{a_{\alpha\omega}\}$ for every $\alpha \in \Lambda^*$, $A = \bigcup_{\omega\in\Lambda_m} A_\omega$ for every $m \in \mathbb{N}^*$ and, more general, $A_\alpha = \bigcup_{\omega\in\Lambda_m} A_{\alpha\omega}$ for every $\alpha \in \Lambda^*$ and every $m \in \mathbb{N}^*$.

(4) The set $\{e_{[\omega]_m} \mid \omega \in \Lambda$ and $m \in \mathbb{N}^*\}$ is dense in A.

(5) The function $\pi : \Lambda \longrightarrow A$ defined by $\pi(\omega) = a_\omega$ is continuous and surjective.

2. Main results

In general the dendrites represent the topological analogue of the trees. Characterizations of dendrite-type attractors can be found in [3] and [5].

Definition 2.1. a) X is *arcwise connected* if for every $x, y \in X$ there exists a continuous function $\varphi : [0, 1] \longrightarrow X$ such that $\varphi(0) = x$ and $\varphi(1) = y$. A continuous function φ as above is called a *path* between x and y. We say that two continuous and injective functions $\varphi, \psi : [0, 1] \longrightarrow X$ are *equivalent* if there exists a function $u : [0, 1] \longrightarrow [0, 1]$ continuous, bijective and increasing such that $\varphi \circ u = \psi$. A class of equivalence is named a *curve*. b) If X is compact, connected and locally connected, then X is called a *dendrite* if for every $x, y \in X$ there exists a unique equivalence class of continuous and injective functions $\varphi : [0, 1] \longrightarrow X$ such that $\varphi(0) = x$ and $\varphi(1) = y$ (i.e. there exists a unique injective curve joining x with y). We remark that two equivalent, continuous and injective functions have the same images. We also consider that the empty set is a dendrite. c) Let $(A_i)_{i\in I}$ be a family of nonempty subsets of X. Then the graph (I, G), where $G = \{(i, j) \mid i, j \in I$ such that $A_i \cap A_j \neq \emptyset$ and $i \neq j\}$ is called the *graph of intersections* associated to the family $(A_i)_{i\in I}$. d) A graph (I, G) is called *connected* if for every $i, j \in I$ there exist $(i_k)_{k=\overline{1,n}} \subset I$ such that $i_1 = i$, $i_n = j$ and $(i_k, i_{k+1}) \in G$ for every $k \in \{1, 2, \ldots, n-1\}$. A family of vertices $(i_1, \ldots, i_m)$ is called a *cycle* if $(i_k, i_{k+1}) \in G$ for every $k \in \{1, \ldots, m\}$ and $i_k \notin \{i_{k+1}, i_{k+2}\}$ for every $k \in \{1, \ldots, m\}$, where by i_{m+1} we understand i_1 and by i_{m+2} we understand i_2. A graph (I, G) is called a *tree* if it is connected and has no cycles.

The following result gives a characterization of dendrites as attractors of some iterated function systems.

Theorem 2.1 ([3]). *Let (X, d) be a complete metric space and $\mathcal{S} = (X, \{f_k\}_{k=\overline{1,n}})$ an iterated function system with $c = \max_{k=\overline{1,n}} Lip(f_k) < 1$. We denote by $A = A(\mathcal{S})$ the attractor of $\mathcal{S}$, by A_k the set $f_k(A)$ for every $k \in \{1, \ldots, n\}$ and by $((1, \ldots, n), G)$ the graph of intersections associated with the family of sets $(A_k)_{k=\overline{1,n}}$. We suppose that the following conditions are true:*

(a) The set $A_i \cap A_j$ is totally disconnected for every $i, j \in \{1, 2, \ldots, n\}$ different.

(b) $A_i \cap A_j \cap A_k = \emptyset$ for every $i, j, k \in \{1, 2, \ldots, n\}$ pairwise different.

(c) $f_k : X \longrightarrow X$ is an injective function for every $k \in \{1, 2, \ldots, n\}$.

Then the following statements are equivalent:

(1) A is a dendrite.

(2) The graph $((1, \ldots, n), G)$ is a tree and $card(A_i \cap A_j) \in \{0, 1\}$ for every $i, j \in \{1, 2, \ldots, n\}$ different.

In [10] and [11] were studied the critical points of an attractor of an iterated function system, some connections with the canonical projection and attractors that are homeomorphic to a quotient space of the shift space by some equivalence relation. We consider the following properties for Hata's tree and Koch's curve: Hata's tree, let us say A, is the attractor of the injective functions defined in the complex plane $f_0(z) = c\bar{z}$ and $f_1(z) = (1 - |c|^2)\bar{z} + |c|^2$, where $c \in \mathbb{C}$ and $|c|, |1 - c| \in (0, 1)$, $A_0 \cap A_1 = f_0(A) \cap f_1(A) = \{|c|^2\}$ and the intersection point satisfies $\{|c|^2\} = \pi(001^\infty) = \pi(10^\infty)$; Koch's curve, let us say B, is the attractor of