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On fractional-Legendre spectral Galerkin method for fractional Sturm–Liouville problems

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ABSTRACT

In this paper, we present a numerical technique for solving fractional Sturm–Liouville problems with variable coefficients subject to mixed boundary conditions. The proposed algorithm is a spectral Galerkin method based on fractional-order Legendre functions. Tedious manipulation of the series appearing in the implementation of the method have been carried out to obtain a system of algebraic equations for the coefficients. Our findings demonstrate the possibility of having no eigenvalues, finite number of eigenvalues or infinite number of eigenvalues depending on the fractional order. The convergence and effectiveness of the present algorithm are demonstrated through several numerical examples.

1. Introduction

The literature reveals that both fractional ODEs and fractional PDEs are very suitable for describing several real-life problems, [1–17]. In recent years, fractional Sturm–Liouville problems has been a subject of numerous investigations due to its importance in many applications in the fields of physics and engineering; especially, the linear fractional partial differential equations [18,19]. It is well-known that, for the case of integer Sturm–Liouville problems, we have infinitely many real eigenvalues. However, the situation is quite different in the fractional Sturm–Liouville problems since we might have either no real eigenvalues, finite number of real eigenvalues or an infinite number of eigenvalues. This result is proven analytically via detailed discussions of different examples in [20–22]. In fact, till now, a significant part of the theory of fractional Sturm–Liouville problems is not fully explored.

Generally speaking, finding the analytical solutions for fractional ODEs or fractional PDEs is far from trivial and often is impossible, see [23–26]. Therefore, many numerical techniques have been designed to solve such type of problems. These methods include Adomian's decomposition method, homotopy perturbation method, variational iteration method, fractional differential transform method, operational matrices techniques based on various orthogonal polynomials and wavelets, nonstandard finite difference method, predictor–corrector approach, spectral methods using fractional Laguerre orthogonal functions and method of lower and upper solutions, the reader is referred to [27–44] and references therein.

In this paper, we consider the following class of fractional Sturm–Liouville problems

$$p(x)D^{\alpha}y(x) + h(x)\,y'(x) + \sigma(x)y(x) + \lambda q(x)y(x) = 0, \quad x \in (0,1), \quad 1 < \alpha \le 2 \tag{1}$$

subject to

$$ay(0) + by'(0) = 0, \quad cy(1) + d\,y'(1) = 0, \tag{2}$$

where $p(x)$, $q(x), p(x), \sigma(x)$ are positive smooth functions on [0,1], and $a, b, c, d \in \mathbb{R}$ with the property $ad - bc \neq 0$. Here, the notation D^{α} denotes the left sided Caputo fractional derivative defined by

$$D^{\alpha}_{0^+}y(x) = \frac{1}{\Gamma(m-\alpha)}\int_0^x (x-t)^{m-\alpha-1}y^{(m)}(t)dt, \quad t > 0, \tag{3}$$

where $m = \lceil \alpha \rceil$ is the smallest integer greater than or equal to α. This problem has received a great amount of interests from many researchers. In fact, finding exact eigenvalues for problem (1) and (2) when p, q or σ is not a constant is extremely difficult. Nevertheless, several numerical and analytical techniques have been used to approximate its solution; see for example [28,45–56]. However, some theoretical results on the fractional Sturm–Liouville problems have been reported in [30,53,55–59]. In addition, Duan et al. [20] and Al-Mdallal et al. [21] have determined the exact eigenvalues for certain classes of fractional Sturm–Liouville problems. They also concluded that the eigenvalues and eigenfunctions might be characterized in terms of the Mittag-Leffler functions. The present work is motivated by the desire to find an approximate so-

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lution of the problem (1) and (2) using fractional-Legendre spectral Galerkin method.

The rest of the paper is organized as follows. A brief review of the fractional calculus and fractional Legendre functions are presented in Section 2. Significant theoretical results are discussed in Section 3. The proposed method is presented in Section 4. The numerical results are presented in Section 5.

2. Preliminary

2.1. Fractional calculus

In this section, we present some essential preliminaries related to fractional calculus theory.

Definition 1. [23,24] The left sided Riemann–Liouville fractional integral operator of order α is defined by

$$J^{\alpha}_{0+}y(x) = \frac{1}{\Gamma(\alpha)}\int_0^x (x-t)^{\alpha-1}y(t)dt \qquad (4)$$

where $y \in L_1[0, T]$, and $\alpha \in \mathbb{R}^+$.

Lemma 1. *[23, 24, 60] Let $\alpha,\ \beta,\ x > 0$ and $\gamma > -1$. Then*

(i) $J^{\alpha}_{0+}J^{\beta}_{0+} = J^{\alpha+\beta}_{0+} = J^{\beta}_{0+}J^{\alpha}_{0+}$

(ii) $J^{\alpha}_{0+}x^{\gamma} = \dfrac{\Gamma(\gamma+1)}{\Gamma(\gamma+1+\alpha)}x^{\gamma+\alpha}$.

[23,24,44] It is well-known that the left sided Caputo fractional derivative (3) is originally defined via the left sided Riemann–Liouville fractional integral (4) as follows

$$D^{\alpha}_{0+}y(x) = J^{m-\alpha}_{0+}y^{(m)}(x), \quad x > 0,$$

where $\alpha \in \mathbb{R}^+$, $m = \lceil \alpha \rceil$ and $y \in L_1[0, T]$, see [61].

Lemma 2. *For $\alpha \in \mathbb{R}^+$, $m = \lceil \alpha \rceil$ and $y \in L_1[0, T]$, we have*

1. $D^{\alpha}_{0+}\ J^{\alpha}_{0+}y(x) = y(x)$.
2. $J^{\alpha}_{0+}\ D^{\alpha}_{0+}y(x) = y(x) - \sum_{k=0}^{m-1} y^{(k)}(0^+)\dfrac{x^k}{k!}$.
3. $D^{\alpha}_{0+}x^r = \begin{cases} \frac{\Gamma(r+1)}{\Gamma(r+1-\alpha)}x^{r-\alpha}, & \text{for } r > \lfloor\alpha\rfloor, \\ 0, & \text{if } r \in \mathbb{N}_0 \ \&\ r \le \lfloor\alpha\rfloor, \\ D.N.E., & \text{if } r \notin \mathbb{N}_0 \ \&\ r < \lfloor\alpha\rfloor, \end{cases}$

where $\lfloor . \rfloor$ is the floor function.

2.2. Fractional-order Legendre functions

It is well-known that the analytical closed form of the shifted Legendre polynomials of degree n is given by

$$L_n(t) = \sum_{k=0}^{n}(-1)^{n+k}\frac{(n+k)!}{(n-k)!(k!)^2}t^k, \quad t \in (0,1). \qquad (5)$$

In [62], the authors were able to generate an orthogonal set of fractional-order Legendre functions based on shifted Legendre polynomials (5) by setting $t = x^{\alpha}$ for $0 < \alpha \le 1$; that is

$$F^{\alpha}_n(x) = \sum_{k=0}^{n}(-1)^{n+k}\frac{(n+k)!}{(n-k)!(k!)^2}x^{k\alpha}. \qquad (6)$$

It can be verified that the functions (6) are particular solutions of the following singular Sturm–Liouville problem

$$\left((x - x^{1+\alpha})F^{\alpha\,\prime}_n(x)\right)' + \alpha^2 n(n+1)x^{\alpha-1}F^{\alpha}_n(x) = 0, \quad x \in (0,1),$$

and $F^{\alpha}_n(0) = (-1)^n$, $F^{\alpha}_n(1) = 1$. Moreover, the function F^{α}_n for $n \in \mathbb{N}$ has the following recursive form

$$F^{\alpha}_{n+1}(x) = \frac{(2n+1)(2x^{\alpha}-1)}{n+1}F^{\alpha}_n(x) - \frac{n}{n+1}F^{\alpha}_{n-1}(x),$$

where

$F^{\alpha}_0(x) = 1,\ F^{\alpha}_1(x) = 2x^{\alpha} - 1.$

In addition, the fractional-order Legendre functions F^{α}_n for $n \in \mathbb{N}$ are orthogonal with respect to the weight function $w(x) = x^{\alpha-1}$ in the interval (0,1); i.e.

$$\int_0^1 F^{\alpha}_n(x)F^{\alpha}_m(x)w(x)dx = \frac{1}{(2n+1)\alpha}\delta_{n,m}. \qquad (7)$$

Using property (3) of Lemma 2; one may easily verify that

$$D^{\alpha}F^{\alpha}_n(x) = \sum_{k=1}^{n}(-1)^{n+k}\frac{(n+k)!}{(n-k)!(k!)^2} \times \frac{\Gamma(k\alpha+1)}{\Gamma((k-1)\alpha+1)}x^{(k-1)\alpha}.$$

Theorem 1. *[62, 63] Let $u \in C[0, 1]$ and $u'(x)$ be a piecewise continuous function on [0,1]. Then, $u(x)$ can be expressed as infinite series; i.e.*

$$u(x) = \sum_{k=0}^{\infty} u_k F^{\alpha}_k(x), \qquad (8)$$

where

$$u_k = (2k+1)\alpha\int_0^1 u(x)F^{\alpha}_k(x)w(x)dx, \qquad (9)$$

and $w(x) = x^{\alpha-1}$.

The next theorem gives the relation between the coefficients of the series solution of $D^{\alpha}u(x)$ and the coefficients of the series expansion of $u(x)$.

Theorem 2. *[63] Let $u \in C[0, 1]$ and $u''(x)$ be a piecewise continuous function on [0,1]. Then, $\sum_{k=0}^{\infty} u^{(\alpha)}_k F^{\alpha}_k(x)$ converges uniformly on [0,1] to $D^{\alpha}u(x)$, $0 < \alpha < 1$, where*

$$u^{(\alpha)}_k = \sum_{j=k+1}^{\infty} a_{jk}u_j, \qquad (10)$$

$$a_{jk} = (2k+1)\alpha\int_0^1 D^{\alpha}F^{\alpha}_j(x)F^{\alpha}_k(x)w(x)dx \qquad (11)$$

for $k = 0, 1, 2, \ldots, j = k+1, k+2, \ldots$.

The explicit form of the product of two Legendre polynomials in terms of Legendre polynomials had been given in 1878 by Neumann and Adams (see Al-Salam [64] for more details) as follows:

$$L_p(x)L_q(x) = \sum_{r=0}^{q}\frac{A_r A_{p-r} A_{q-r}}{A_{p+q-r}}\frac{2p+2q-4r+1}{2p+2q-2r+1}L_{p+q-2r}(x), \quad p \ge q, \qquad (12)$$

where $A_r = \dfrac{\prod_{i=0}^{r-1}\frac{1}{2}+i}{r!}$. For simplicity, we set

$$B^{p,q}_r = \frac{A_r A_{p-r} A_{q-r}}{A_{p+q-r}}\frac{2p+2q-4r+1}{2p+2q-2r+1}. \qquad (13)$$

Notice that, the following properties of $B^{p,q}_r$ hold true for any $r, p, q, i, j \in \mathbb{N}$.

1. $B^{p,q}_r = 0$ for either r, $p-r$ or $q-r$ is negative.
2. $B^{i,j}_r = B^{j,i}_r$.

As a result of the above mentioned properties, the condition $p \ge q$ in (14) can be dropped. It can be easily shown that the expression form (12) should satisfy the product of two fractional-Legendre functions, that is

$$F^{\alpha}_p(x)F^{\alpha}_q(x) = \sum_{r=0}^{q} B^{p,q}_r F^{\alpha}_{p+q-2r}(x). \qquad (14)$$