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Efficient numerical approach for solving fractional partial differential equations with non-singular kernel derivatives

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ABSTRACT

Adams-Bashforth was recognized as powerful numerical method to solve linear and non-linear ordinary differential equations. Nevertheless the method was applicable only for ordinary differential equations mostly with integer order. Atangana and Batogna have extended this method for partial differential equation with the Atangana-Baleanu fractional derivative. In this paper, to accommodate partial differential equation with Caputo-Fabrizio derivative, we suggest the corresponding method with this derivative. We applied the method to solve numerically a very interesting non-linear partial differential equation accounting for the motion of a viscous fluid. Some simulations are presented to test the efficiency of the numerical method.

1. Introduction

In the last decades, numerical methods have been used very successfully to solve complex mathematical models arising in all fields of science, technology and engineering, especially those mathematical models that could not be solved using analytical methods. As the science, technology and industries evolved, mankind, each day discovers new and complex real world problems that need urgent attention. In many instances, already suggested numerical schemes are found not suitable for solving accurately such complicated mathematical models, therefore new schemes are needed. One of the powerful numerical scheme for solving nonlinear equations is perhaps the Adams-Bashforth method which is a multi-steps methods [5–10], the method has been intensively applied in many problems arising in chemistry, biology, epidemiology and engineering in dynamical systems, physics, earth science, chemistry) and engineering disciplines (such as computer science, electrical engineering), as well as in the social sciences (such as economics, psychology, sociology, political science). However, this method can only be used in the case of nonlinear ordinary differential equations, an extension was also made for fractional differential equations. Only very recently, Atangana and Batogna extended this method to solve partial differential equations, in particular those partial differential equation with the Atangana-Baleanu derivative. The method has attracted attention of some researchers and has been applied with great success. Nevertheless there exists in the literature another important differential operator with no singular kernel known as Caputo-Fabrizio derivative, which naturally has generated new mathematical models constructed with partial and ordinary differential equations [12,16–23]. In this paper in order to solve such mathematical models, a corresponding numerical method to what was proposed by Atangana and Batogna in the case of Atangana-Baleanu derivative is needed and will be derived in this paper. The method developed by Atangana and Batogna is a combination of Laplace transform operator, Adams-Bashforth numerical scheme, inverse Laplace transformation and finite difference method like backward and forward.

The remainder of this paper is follows that , in section one; some useful definitions and properties of Caputo-Fabrizio and Atangana-Baleanu derivative are given, in section two; general fractional differential equation with fading memory included via Caputo-Fabrizio fractional derivative is considered and numerical scheme for equation is obtained. In the subsection of two, application of the Atangana-Batogna method on the time fractional motion of a viscous fluid equation via Caputo-Fabrizio derivative is done. In section three; general fractional differential equation with fading memory included via Atangana-Baleanu fractional derivative is considered and numerical scheme for equation is obtained. Similarly of the subsection two, in the subsection of three, application of the Atangana-Batogna method on the time fractional motion of a viscous fluid equation via Atangana-Baleanu derivative is done. Finally in section five, we present the numerical simulation of the our equation for different values.

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2. Fractional derivatives with exponential and Mittag-Leffler kernels

In the last 3 years, the field of fractional differentiation and integration has witness a great development as some new and very promising differential and integral operators have been suggested". These new differential and integral operators have been used intensively and are in fashion nowadays. In this section, let us remind the definitions of the new fractional derivatives [1–3].

Definition 1. Let $f \in H^1(a, b)$, $b > a$, $\alpha \in [0, 1]$ then, the definition of the new fractional derivative (Atangana-Baleanu derivative in Caputo sense) is given as:

$$ {}_{a}^{ABC}D_t^{\alpha} f(t) = \frac{B(\alpha)}{1-\alpha} \int_a^t f'(x) E_{\alpha}\left[-\alpha \frac{(t-x)^{\alpha}}{1-\alpha}\right] dx, \tag{1}$$

where ${}_{a}^{ABC}D_t^{\alpha}$ is fractional operator with Mittag-Leffler kernel in the Caputo sense with order α with respect to t and $B(\alpha) = B(0) = B(1) = 1$ is a normalization function [3].

It can be noted that the above definition is helpful to model real world problems. Also it has a great advantage while using the Laplace transform to solve problem with initial condition.

Definition 2. Let $f \in H^1(a, b)$, $b > a$, $\alpha \in [0, 1]$ and not differentiable then, the definition of the new fractional derivative (Atangana-Baleanu fractional derivative in Riemann-Liouville sense) is given as:

$$ {}_{a}^{ABR}D_t^{\alpha} f(t) = \frac{B(\alpha)}{1-\alpha} \frac{d}{dt} \int_a^t f(x) E_{\alpha}\left[-\alpha \frac{(t-x)^{\alpha}}{1-\alpha}\right] dx. \tag{2}$$

Definition 3. The fractional integral of order α of a new fractional derivative is defined as:

$$ {}_{a}^{AB}I_t^{\alpha} f(t) = \frac{1-\alpha}{B(\alpha)} f(t) + \frac{\alpha}{B(\alpha)\Gamma(\alpha)} \int_a^t f(y)(t-y)^{\alpha-1} dy. \tag{3}$$

When α is zero, initial function is obtained and when α is 1, the ordinary integral is obtained.

Theorem 1. The following time fractional ordinary differential equation

$$ {}_{0}^{ABC}D_t^{\alpha} f(t) = u(t), \tag{4}$$

has a unique solution with taking the inverse Laplace transform and using the convolution theorem below [4]:

$$ f(t) = \frac{1-\alpha}{B(\alpha)} u(t) + \frac{\alpha}{B(\alpha)\Gamma(\alpha)} \int_a^t u(y)(t-y)^{\alpha-1} dy. \tag{5}$$

Definition 4. : Let $f \in H^1(a, b)$, $b > a$, $\alpha \in [0, 1]$ then, the new Caputo derivative of fractional derivative is defined as :

$$ {}_{a}^{CF}D_t^{\alpha} f(t) = \frac{M(\alpha)}{1-\alpha} \int_a^t f'(x) \exp\left[-\alpha \frac{(t-x)}{1-\alpha}\right] dx. \tag{6}$$

where $M(\alpha)$ is a normalization function such that $M(0) = M(1) = 1$. However, if the function does not belongs to $H^1(a, b)$ then, the derivative can be reformulated as

$$ {}_{a}^{CF}D_t^{\alpha} f(t) = \frac{\alpha M(\alpha)}{1-\alpha} \int_a^t (f(t) - f(x)) \exp\left[-\alpha \frac{(t-x)}{1-\alpha}\right] dx. \tag{7}$$

Definition 5. Let $0 < \alpha < 1$. The fractional integral of order α of a function f is defined by

$$ I_{\alpha}^{t} f(t) = \frac{2(1-\alpha)}{(2-\alpha)M(\alpha)} f(t) + \frac{2\alpha}{(2-\alpha)M(\alpha)} \int_0^t f(s) ds, \ t \geq 0. \tag{8}$$

3. Numerical method for partial differential equation via Caputo-Fabrizio derivative

Due to the wider applicability of the new fractional differential operator called the Caputo-Fabrizio derivative, some new techniques including analytical and numerical are requested to accommodate mathematical models constructed via this operators. While it is considered as local in terms of time memory, but yet the differential operator is in-build with another kind of non-locality. The kind of non-locality of this operator can be seeing in details in the following distinguished already published papers [11,13,14] . We aim in this section, to provide a numerical scheme that was already suggested by Atangana and Batogna for solving partial differential equations with non-integer order [15]. To do this, we present the derivation using a general partial differential equation with the Caputo-Fabrizio derivative.

$$ {}_{0}^{CF}D_t^{\alpha} u(x, t) = Au(x, t) + Bu(x, t), \tag{9}$$

where A is a linear operator and B is a non-linear operator. Applying Laplace transform on both sides of the equation we have

$$ \begin{aligned} \mathcal{L}\left\{ {}_{0}^{CF}D_t^{\alpha} u(x, t)\right\} &= \mathcal{L}\{Au(x, t) + Bu(x, t)\}, \\ {}_{0}^{CF}D_t^{\alpha} u(p, t) &= \mathcal{L}\{Au(x, t) + Bu(x, t)\}. \end{aligned} \tag{10}$$

Let us take $u(p, t) = u(t)$ and $\mathcal{L}\{Au(x, t) + Bu(x, t)\} = F(u, t)$ then we have

$$ {}_{0}^{CF}D_t^{\alpha} u(t) = F(u, t) \tag{11}$$

then we have

$$ \frac{M(\alpha)}{1-\alpha} \int_0^t u'(x) \exp\left[-\alpha \frac{t-x}{1-\alpha}\right] dx = F(u, t). \tag{12}$$

Using the fundamental theorem of calculus, we convert the above to Caputo Fabrizio fractional integral equation as below:

$$ u(x, t) - u(x, 0) = \frac{2(1-\alpha)}{(2-\alpha)M(\alpha)} F(u, t) + \frac{2\alpha}{(2-\alpha)M(\alpha)} \int_0^t F(u, \tau) d\tau. \tag{13}$$

When $t = t_{n+1}$, we have

$$ u(x, t_{n+1}) - u(x, 0) = \frac{2(1-\alpha)}{(2-\alpha)M(\alpha)} F(u, t_{n+1}) + \frac{2\alpha}{(2-\alpha)M(\alpha)} \int_0^{t_{n+1}} F(u, \tau) d\tau. \tag{14}$$

When $t = t_n$, we have

$$ u(x, t_n) - u(x, 0) = \frac{2(1-\alpha)}{(2-\alpha)M(\alpha)} F(u, t_n) + \frac{2\alpha}{(2-\alpha)M(\alpha)} \int_0^{t_n} F(u, \tau) d\tau. \tag{15}$$

Then we can write follows that

$$ u(x, t_{n+1}) - u(x, t_n) = \frac{2(1-\alpha)}{(2-\alpha)M(\alpha)} (F(u, t_{n+1}) - F(u, t_n)) + \frac{2\alpha}{(2-\alpha)M(\alpha)} \left(\int_0^{t_{n+1}} F(u, \tau) d\tau - \int_0^{t_n} F(u, \tau) d\tau \right). \tag{16}$$

Here,

$$ \begin{aligned} \int_0^{t_{n+1}} F(u, \tau) d\tau &= \sum_{j=0}^{n} \int_{t_j}^{t_{j+1}} F(u, \tau) d\tau, \\ \int_0^{t_n} F(u, \tau) d\tau &= \sum_{j=0}^{n-1} \int_{t_j}^{t_{j+1}} F(u, \tau) d\tau. \end{aligned} \tag{17}$$