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Chaos in a 5-D hyperchaotic system with four wings in the light of non-local and non-singular fractional derivatives

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ABSTRACT

A new 5-D hyperchaotic system with four wings is studied in the light of the newly introduced operator by Atangana and Baleanu with non-local and non-singular fading memory. The basic properties and stability analysis are studied. Picard–Lindelof method is used to examine the existence and uniqueness of solutions of the new 5-D hyperchaotic system with four wings. The numerical simulation results depict a new chaotic behaviours with the ABC numerical scheme.

1. Introduction

Chaotic models have been widely studied in the three well-known approaches: frequency domain method (FDM) [1,2], Adams-Bashforth-Moulton algorithm (ABM) [3,4] and Adomian decomposition method (ADM) [5–7] A new aspect of chaotic system called hyperchaotic system was first proposed by Rossler, then several versions have been constructed and reported [8–10]. In recent times, Zarei proposed a new 5-D hyper chaotic system with four wings [11], with multi-parameter, multi-wing which had been studied in both integer and non-integer version.

Mathematical modeling has become a powerful tool that helps researchers to predict the evolution of many phenomena. The most common instrument used is the concept of differentiation and integration. There have been two classes of operators established by researchers namely, the classical and non-local differential operators. The non-local operator has attracted many researchers because of its ability to model complex phenomena and providing accurate predictions [12–14]. Fractional derivatives become excellent instrument for the description of memory and hereditary properties of various materials and processes. Such effects are neglected in models with classical integer-order. This can be viewed as the main advantage of fractional derivatives. It also plays a crucial role in the description of dynamics between two different points in many other fields [15–17]

The non-local operators differentiation hinge on many laws and the prominent among them are the power law, exponential decay law and generalized Mittag-Leffler law. In order to differentiate these laws some names have been linked to them and these include Riemann-Liouville and Caputo is responsible for power law or non-local and singular kernel aspect, Caputo–Fabrizio deals with non-singular and local kernel while Atangana–Baleanu is associated with non-local and non-singular aspect. It has been found out that the kernel Mittag-Leffler function has more general character compare with the other two and one can say that they are special cases of Atangana–Baleanu fractional operator [18–20].

The Atangana–Baleanu and Caputo–Fabrizio operators have unique characteristics in the mean square displacement while Riemann–Liouville possesses scale invariant. Their respective probability distributions are from Gaussian to non-Gaussian crossover. The Fabrizio kernel has equilibrium between the transition while the Atangana–Baleanu kernel also has a crossover for waiting time distribution which stretches exponential to power law [20,21]. This property therefore, makes Atanagan–Belaenu fractional derivative operator very powerful and more preferred.

The aim of this study is to use Atangana–Beleanu fractional derivative operator to explore new attractors with the new 5-D hyperchaotic system with four wings and also to derive the uniqueness and existence of solutions of the model.

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2. Fractional order derivative with Mittag-Leffler kernel

In this section, some definitions and properties of the fractional derivative are given via non-singular and non-local Mittag-Leffler kernel [6,8].

Definition 2.1. *Consider* $(a,\ b) > a,\ \alpha \in [0,\ 1]$ *then, the Atangana–Baleanu derivative in Caputo sense is defined as*

$$ {}_{a}^{ABC}D_t^{\alpha}(f(t)) = \frac{B(\alpha)}{1-\alpha}\int_a^t f^1(x)E_\alpha\left[-\alpha\frac{(t-x)^\alpha}{1-\alpha}\right]dx \tag{1}$$

where $M(\alpha)$ *is a normalization function and* $B(0) = B(1) = 1$ [5,8].

Definition 2.2. *D Consider* $f \in H^1(a,\ b) > a,\ b > a,\ \alpha \in [0,\ 1]$ *and not differentiable then, the Atangana–Baleanu fractional derivative in Riemann–Liouville sense is defined as*

$$ {}_{a}^{ABR}D_t^{\alpha}(f(t)) = \frac{B(\alpha)}{1-\alpha}\frac{d}{dt}\int_a^t f(x)E_\alpha\left[-\alpha\frac{(t-x)^\alpha}{1-\alpha}\right]dx \tag{2}$$

Definition 2.3. *The fractional integral of order* α *of a new fractional derivative is defined as*

$$ {}_{a}^{AB}I_t^{\alpha}\{f(t)\} = \frac{1-\alpha}{B(\alpha)}f(t) + \frac{\alpha}{B(\alpha)\Gamma(\alpha)}\int_a^t f(y)(t-y)^{\alpha-1}dy \tag{3}$$

When α *is zero, the initial function is obtained and when* α *is 1, the ordinary integral is derived.*

It is obvious, from many real life that applications based on the above operator provide accurate prediction. In this regard, we present some relations between the derivative and Laplace transform. The relationship between the operator and the Laplace transform operator are given by as $n = 1$:

$$ L\left\{{}_{0}^{ABR}D_t^{\alpha}f(t)\right\}(q) = \frac{B(\alpha)}{1-\alpha}\frac{q^\alpha L\{f(t)\}(q)}{q^\alpha + \dfrac{\alpha}{1-\alpha}}$$

$$ L\left\{{}_{0}^{ABC}D_t^{\alpha}f(t)\right\}(q) = \frac{B(\alpha)}{1-\alpha}\frac{q^\alpha L\{f(t)\}(q) - q^{\alpha-1}f(0)}{q^\alpha + \dfrac{\alpha}{1-\alpha}} \tag{4}$$

Theorem 1. *[8, 9] Take into Consideration f as a continuous function on a closed interval [a, b] then the following inequality is derived on [a, b] expressed as:*

$$ \left\|{}_{0}^{ABR}D_t^{\alpha}f(t)\right\| < \frac{B(\alpha)}{1-\alpha}\|f(x)\| \tag{5}$$

Here, $\|f(t)\| = \max_{a\le x\le b}\|f(t)\|$ The Lipschitz condition is satisfied by (ABR) and (ABC) derivatives and the condition as follows:

$$ \left\|{}_{0}^{ABR}D_t^{\alpha}g(t) - {}_{0}^{ABR}D_t^{\alpha}f(t)\right\| \le H\|g(t)-f(t)\| \quad \textit{and}$$
$$ \left\|{}_{0}^{ABC}D_t^{\alpha}g(t) - {}_{0}^{ABC}D_t^{\alpha}f(t)\right\| \le H\|g(t)-f(t)\| \tag{6}$$

Extensive details on the proof of the theorems above can be found in [23]. The Mittag-Leffler function consists of one and two parameters $E_\alpha(z)$, $E_{\alpha,\beta}(z)$ is expressed as [22]

$$ E_\alpha(z) = \sum_{k=0}^{\infty}\frac{z^k}{\Gamma(\alpha k+1)},\quad z\in\mathbb{C},\quad \Re(\alpha) > 0, \tag{7}$$

$$ E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty}\frac{z^k}{\Gamma(\alpha k+\beta)},\quad z,\ \alpha,\ \beta\in\mathbb{C},\quad \Re(\alpha) > 0,$$
$$ \textit{and}\quad \Re(\beta) > 0, \tag{8}$$

where $\Gamma(\cdot)$ denotes the Gamma function, and $e^z = E_1(z)$.

3. Mathematical Model Formulation

In this work, a modified version of hyperchaotic system with four wings in [4] was examined. The equations of the integer-order 5-D hyperchaotic system with four wings are

$$\begin{cases} \frac{dx_1}{dt} = -ax_1(t) + x_2(t)x_3(t) \\ \frac{dx_2}{dt} = -bx_2(t) + fx_5(t) \\ \frac{dx_3}{dt} = -cx_3(t) + gx_4(t) + x_1(t)x_2(t) \\ \frac{dx_4}{dt} = dx_4(t) - hx_3(t) + ax_1(t) \\ \frac{dx_5}{dt} = ex_5(t) - x_2(t)x_1^2(t) \end{cases} \tag{9}$$

where *a*, *b*, *c*, *d*, *e*, *f*, *g* and *h* are system parameters. With only one equilibrium point, the system can generate complex hyperchaotic attractors with four wings, and it has two typical parameter sets. Change the above system equations to the following fractional-order form:

4. Non-negative solutions

In this section, we prove the positivity of the solutions of model (9). Let $R_+^5 = \left\{v\in R^3 | v\ge 0\right\}$ and $v(t) = (x_1(t), x_2(t), x_3(t), x_4(t), x_5(t))^T$

To prove the actual theorem, we require the following generalized mean value theorem [9] and corollary

Lemma 4.1. *([10]) Suppose that* $f(v)\in C[a,\ b]$ *and* $D_a^\alpha f(v)\in C[a, b]$, *for* $0 < \alpha \le 1$ *then we have*

$$ f(u) = f(a) + \frac{1}{\Gamma(\alpha)}(D_a^\alpha f)(\varepsilon)(v-a)^\alpha \tag{10}$$

with $a \le \varepsilon \le v$, $\forall v\in(a,\ b]$ *and T(.) is the gamma function.*

Corollary 4.1. *Suppose that* $f(v)\in C[a,\ b]$ *and* $D_a^\alpha f(v)\in C[a, b]$, *for* $0 < \alpha \le 1$. *If* $D_a^\alpha f(v) \ge 0$, $\forall v\in(a,\ b)$ *then f(v) is non-decreasing for each* $v\in[a,\ b]$. *If* $D_a^\alpha \le 0$, $\forall v\in(a,\ b)$ *then f(v) is non-increasing for each* $v\in[a,\ b]$.

We now determine the proof of the main theorem.

Theorem 2. *There is a unique solution* $u(t) = (x(t), y(t), z(t))^T$ *to model (1) on* $t\ge 0$ *and the solution will stay in* R_+^3.

Proof from Theorem 3.1 and Remark 3.2 of [9], we see that the solution on $(0, +\infty)$ of the initial value problem exists and at the same time unique. We now establish that the non-negative orthant R_+^5 is a positively invariant region. In order to carry out this, we require to show that on each hyperplane bounding the non-negative orthant, the vector field points to R_+^5 . From model (9), we determine that

$$\begin{cases} Dx_1|_{x_1=0} = x_2(t)x_3(t) \\ Dx_2|_{x_2=0} = fx_5(t) \\ Dx_3|_{x_3=0} = gx_4(t) + x_1(t)x_2(t) \\ Dx_4|_{x_4=0} = dx_4(t) + ax_1(t) \\ Dx_5|_{x_5=0} = ex_5(t) \end{cases} \tag{11}$$

Thus, by Corollary 2, the solution of model (9) will remain in R_+^5 .