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On the quasi-normal modes of a Schwarzschild white hole for the lower angular momentum and perturbation by non-local fractional operators

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ABSTRACT

We investigate conditions for the quasi-normal modes of a Schwarzschild white hole for lower angular momentum. In determining these normal modes, we use numerical methods to solve the solution of the linearized Einstein vacuum equations in null cone coordinates. The same model is generalized to non-local fractional operator theory where the model is solved numerically thanks to a method proposed by Toufik and Atangana. In fact, approaching this kind of problem analytically seems to be an impossible task as comprehensively articulated in the literature. We show existence of quasi-normal modes of a Schwarzschild white hole for lower angular momentum $l = 2$. Moreover, the non-local fractional operator appears to be a perturbator factor for the system as shown by numerical simulations that compare the types of dynamics in the system.

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1. Introduction

The linear perturbation theory applied to black hole was developed many years ago. The main idea here comes from the fact that the vacuum Einstein equations are linearized about the Schwarzschild (or Kerr) geometry which is described by the standard well-known coordinates, namely (t, r, θ, ϕ). Then, one can apply a simple separation of variables ansatz, leading to the metric quantities that behave like an unknown function of variable $r \times Y_{\ell m}(\theta, \phi)\exp(i\sigma t)$. It is important to mention that the angular dependence is somewhat very complicated, and the technical details related to it is available in the literature. One of the common ways to obtain the quasi-normal modes is by seeking and investigating the solutions to the Zerilli equation which should verify suitable boundary conditions in the event horizon's neighborhood and infinity's neighborhood. It was then proved that there exist solutions only in the case of certain special values taken by the parameter σ [2,5,10].

The theory of quasi-normal mode has slowly become a cornerstone for the modern theory of general relativity. It appears in some kind of simulations, like the numerical relativity simulations of binary black hole coalescence. And, while it is not yet actually observed, we really expect it to be measured by the LIGO collaboration, and certainly by LISA, and therefore, leading to concise information about the parameters that describe a black hole from some coalescence event.

The approach commonly used for linear perturbations of a black hole includes the performance of linearization using well-known standard Schwarzschild (or Kerr) coordinates (t, r, θ, ϕ). We can also perform the same linearization making use of Bondi–Sachs coordinates, which represent a system of coordinates based on outgoing null cones. Those techniques have been done in previous works and the main objective was find analytic solutions of the linearized Einstein equations, and this, for the sole goal of testing numerical relativity codes [9,15,19]. As with the standard well-known approach, we finish by obtaining a second order ordinary differential equation involving ℓ and σ as parameters, Eq. (2.9). However, after the quasi-normal modes were found for that equation, it was pointed out that they are different from those found in Zerilli equation. The reason come from the fact that the different physical problems are considered in the two cases, as

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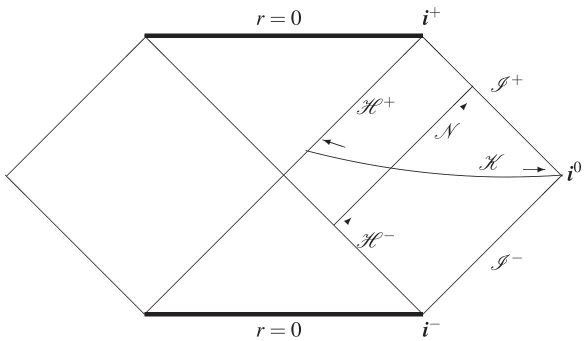

Fig. 1. Carter–Penrose conformal diagram for a Schwarzschild geometry.

illustrated in the Carter–Penrose conformal diagram of Schwarzschild space-time (Fig. 1) below [5]. In Fig, reffig1, $\mathcal{K}$ is a typical hypersurface used in finding the quasi-normal modes of the Zerilli equation, and the direction of wave propagation at the boundaries of $\mathcal{K}$ is shown by arrows. On the other hand, N is a typical hypersurface used in finding the quasi-normal modes of Eq. (2.9). From the direction of wave propagation on N, it is clear that this leads to the quasi-normal modes of a white holes. To briefly summarize the content of this paper, we start with the background material work on the Bondi–Sachs metric and linearized solutions within the Bondi frame. After that, we describe the mathematical algorithm and the numerical approach to calculating the quasi-normal modes of a Schwarzschild white hole, and continue by presenting the results. Then follows the generalization to non-local operator theory and numerical simulations.

2. Background material

The Bondi–Sachs formalism uses coordinates $x^i = (u, r, x^A)$ based upon a family of outgoing null hypersurfaces. We label these hypersurfaces by $u = const.$, null rays by x^A $(A = 2, 3)$, and the surface area coordinate by r. In this coordinates system the Bondi–Sachs metric [6,18] takes the form

$$ds^2 = -\left[e^{2\beta}\left(1+\frac{W}{r}\right) - r^2 h_{AB}U^A U^B\right]du^2 - 2e^{2\beta}dudr - 2r^2 h_{AB}U^B dudx^A + r^2 h_{AB}dx^A dx^B, \tag{2.1}$$

where $h^{AB}h_{BC} = \delta^A_B$ and $det(h_{AB}) = det(q_{AB})$, with q_{AB} being a unit sphere metric, U is the spin-weighted field given by $U = U^A q_A$. For a Schwarzschild space-time, $W = -2M$. We define the complex quantity J by

$$J = q^A q^B h_{AB}/2. \tag{2.2}$$

For the Schwarzschild space-time, we have J and U being zero and thus they can be regarded as a measure of the deviation from spherical symmetry, and in addition, they contain all the dynamic content of the gravitational field in the linearized regime [4]. Usually we can describe this space-time by $\beta = 0$ and $W = -2M$, or by $\beta = \beta_c$(constant) and $W = (e^{2\beta_c} - 1)r - 2M$.

For spherical harmonics we use $_sZ_{lm}$ rather than $_sY_{lm}$ as basis functions as follows [3]

$$\begin{aligned} {}_sZ_{lm} &= \frac{1}{\sqrt{2}}[{}_sY_{lm} + (-1)^m{}_sY_{l-m}] \quad \text{for} \quad m > 0 \\ {}_sZ_{lm} &= \frac{i}{\sqrt{2}}[(-1)^m{}_sY_{lm} -{}_s Y_{l-m}] \quad \text{for} \quad m > 0 \\ {}_sZ_{l0} &= {}_sY_{l0}, \end{aligned} \tag{2.3}$$

The $s = 0$ will be omitted in the case $s = 0$, i.e. $Z_{lm} ={}_0 Z_{lm}$. The $_sZ_{lm}$ are orthonormal and real. We assume the following ansatz

$$J = Re(J_0(r)e^{i\sigma u})_2Z_{lm}, \quad U = Re(U_0(r)e^{i\sigma u})_2Z_{lm},$$

$$\beta = Re(\beta_0(r)e^{i\sigma u})_2Z_{lm}, \quad w = Re(w_0(r)e^{i\sigma u})_2Z_{lm}, \tag{2.4}$$

where l, r_0, and σ are fixed. The Einstein vacuum equations for the hypersurface equations and evolution equation are [3]

$$R_{11} : \frac{4}{r}\beta_{,r} = 8\pi T_{11} \tag{2.5}$$

$$q^A R_{1A} : \frac{1}{2r}(4\eth\beta - 2r\eth\beta_{,r} + r\bar{\eth}J_{,r} + r^3U_{,rr} + 4r^2U_{,r}) = 8\pi q^A T_{1A} \tag{2.6}$$

$$h^{AB}R_{AB} : (4 - 2\eth\bar{\eth})\beta + \frac{1}{2}(\bar{\eth}^2 J + \eth^2\bar{J}) + \frac{1}{2r^2}(r^4\eth\bar{U} + r^4\bar{\eth}U)_{,r} - 2\omega_{,r} = 8\pi\,(h^{AB}T_{AB} - r^2T) \tag{2.7}$$

$$q^A q^B R_{AB} : -2\eth^2\beta + (r^2\eth U)_{,r} - 2(r-M)J_{,r} - \left(1 - \frac{2M}{r}\right)r^2 J_{,rr} + 2r(rJ)_{,ur} = 8\pi q^A q^B T_{AB}. \tag{2.8}$$

By applying ansatz (2.4) to equations (2.7)–(2.8) we get system of ordinary differential equations which we solve to get

$$x^3(1-2xM)\frac{d^2J_2}{dx^2} + 2\frac{dJ_2}{dx}(2x^2 + i\sigma x - 7x^3M) - 2(x(l^2 + l - 2)/2 + 8Mx^2 + i\sigma)J_2 = 0 \tag{2.9}$$

where $J_2(x) \equiv d^2J_{0+}/dx^2$ and $x = 1/r$. Eq. (2.9) cannot be solved analytically, so, we are going to solve it numerically.

2.1. The transformation to Riccati equation

From Eq. (2.9) with $M = 1$ we get

$$x^3(1-2x)\frac{d^2J_2}{dx^2} + 2x(2x + i\sigma - 7x^2)\frac{dJ_2}{dx} - 2(2x + 8x^2 + i\sigma)J_2 = 0. \tag{2.10}$$

Now our task is to solve Eq. (2.9) in the interval [0,0.5]. To obtain stability of the numerics, and hence better numerical results, Chandrasekhar et al. [7] showed that Eq. (2.9) is better handled as a Riccati type equation. To do this we start by making the transformation

$$J_2(x) \;\rightarrow\; u(x) = \frac{1}{J_2(x)}\frac{dJ_2(x)}{dx}, \tag{2.11}$$

so that

$$\frac{dJ_2}{dx} = uJ_2, \tag{2.12}$$

and thus

$$\frac{d^2J_2}{dx^2} = \frac{du}{dx}J_2 + u\frac{dJ_2}{dx} = J_2\left(\frac{du}{dx} + u^2\right). \tag{2.13}$$

By substituting Eq. (2.13) into Eq. (2.10) we get

$$x^3(1-2x)\left(\frac{du}{dx} + u^2\right) + 2x(2x + i\sigma - 7x^2)u - 2(2x + 8x^2 + i\sigma) = 0. \tag{2.14}$$

We further make another transformation

$$u(x) \;\rightarrow\; v(x) = \frac{1}{u(x)}, \tag{2.15}$$

and we observe that at small x, $v \sim x$. Now substituting Eq. (2.15) into Eq. (2.14) we obtain

$$x^3(1-2x)\left(1 - \frac{dv}{dx}\right) + 2x(2x + i\sigma - 7x^2)v$$

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