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On the solution of time-fractional KdV–Burgers equation using Petrov–Galerkin method for propagation of long wave in shallow water

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ABSTRACT

In the present article, Petrov–Galerkin method has been utilized for the numerical solution of nonlinear time-fractional KdV–Burgers (KdVB) equation. The nonlinear KdV–Burgers equation has been solved numerically through the Petrov–Galerkin approach utilising a quintic B-spline function as the trial function and a linear hat function as the test function . The numerical outcomes are observed in good agreement with exact solutions for classical order. In case of fractional order, the numerical results of KdV–Burgers equations are compared with those obtained by new method proposed in [1]. Numerical experiments exhibit the accuracy and efficiency of the approach in order to solve nonlinear dispersive and dissipative problems like the time-fractional KdV–Burgers equation.

1. Introduction

A well-known equation viz. the Korteweg–de Vries–Burgers equation has been examined in the present study which plays an essential role in both applied mathematics and physics. The analysis of nonlinear phenomena has always been an active subject in applied science and physics. In past few years, tremendous effort has been anticipated on the analysis of nonlinear evolution equations occurred in mathematical physics. As a classic nonlinear PDE, the KdV type equations [2–4] had been received more attention particularly due to its diverse implementations in plasma physics, solid state physics and quantum field theory.

The KdV–Burgers equation is a nonlinear partial differential equation of the form

$$u_t + \varepsilon u u_x - \nu u_{xx} + \mu u_{xxx} = 0 \tag{1.1}$$

which was first derived by Su and Gardner [5]. It arises in quite a lot of contexts as a model equation incorporating a few foremost physical phenomena viz. dispersion, viscosity and nonlinear advection. This equation arises within the description of long wave propagation in shallow water [6], propagation of waves in elastic tube stuffed with a viscous fluid [7] and weakly nonlinear plasma waves with certain dissipative effects [8]. It additionally represents long wavelength approximations where the effect of the nonlinear advection uu_x is counterbalanced by means of the dispersion u_{xxx}.

Eq. (1.1) is combination of the KdV equation (when $\nu = 0$) and the Burgers' equation (when $\mu = 0$). The KdV equation was first proposed by Korteweg and Vries in 1895 [9]. Initially this equation is derived as an evolution equation that governs small amplitude, long surface gravity waves propagating in a shallow channel of water [10]. This equation was also utilized to analyze the change in shape of long waves moving in a rectangular channel [9,11]. In the year 1939, Burger proposed an equation (known as Burgers' equation) for the study of turbulence and approximate theory of flow through a shock wave traveling in a viscous fluid [12–14]. When diffusion dominates dispersion, the numerical solutions of Eq. (1.1) have a tendency to behave like Burgers' equation solutions and hence the steady-state solutions of the KdVB equation are monotonic shocks. However, when dispersion dominates, the KdV behaviour is observed and the shocks are oscillatory.

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Table 1
The absolute errors acquired by Petrov–Galerkin method with respect to solution attained by new method in Ref. [1] for time-fractional KdV–Burgers equation at various points of x and t taking $\alpha = 0.75$, $\lambda = 1$, $\varepsilon = 6$, $\nu = 1$ and $\mu = 2$.

x	$\lvert u_{PGM} - u_{Exact} \rvert$							
	$t = 0.2$	$t = 0.3$	$t = 0.4$	$t = 0.5$	$t = 0.6$	$t = 0.7$	$t = 0.8$	$t = 0.9$
0.1	1.46833E-3	2.79680E-3	4.03999E-3	5.21834E-3	6.34158E-3	7.41499E-3	8.44173E-3	9.42393E-3
0.2	1.47268E-3	2.80387E-3	4.04863E-3	5.22763E-3	6.35075E-3	7.42339E-3	8.44882E-3	9.42925E-3
0.3	1.47692E-3	2.81075E-3	4.0570E-3	5.23656E-3	6.35949E-3	7.43129E-3	8.45536E-3	9.43397E-3
0.4	1.48107E-3	2.81745E-3	4.06509E-3	5.24514E-3	6.36780E-3	7.43871E-3	8.46143E-3	9.43807E-3
0.5	1.48512E-3	2.82396E-3	4.07291E-3	5.25337E-3	6.37568E-3	7.44562E-3	8.46676E-3	9.44157E-3
0.6	1.48908E-3	2.83028E-3	4.08045E-3	5.26124E-3	6.38314E-3	7.45204E-3	8.47164E-3	9.44446E-3
0.7	1.49293E-3	2.83640E-3	4.08772E-3	5.26875E-3	6.39017E-3	7.45797E-3	8.47595E-3	9.44674E-3
0.8	1.49669E-3	2.84234E-3	4.09472E-3	5.27591E-3	6.39676E-3	7.46341E-3	8.47972E-3	9.44842E-3
0.9	1.50034E-3	2.84809E-3	4.10143E-3	5.28272E-3	6.40293E-3	7.46834E-3	8.48293E-3	9.44949E-3
1.0	1.50390E-3	2.85364E-3	4.10787E-3	5.28916E-3	6.40867E-3	7.47279E-3	8.48559E-3	9.44996E-3

Table 2
The absolute errors acquired by Petrov–Galerkin method with respect to solution attained by new method in Ref. [1] for time-fractional KdV–Burgers equation at various points of x and t taking $\alpha = 0.5$, $\varepsilon = 6$, $\nu = 0.05$, $\lambda = 10$ and $\mu = 0.1$.

x	$\lvert u_{PGM} - u_{Exact} \rvert$								
	$t = 0.1$	$t = 0.2$	$t = 0.3$	$t = 0.4$	$t = 0.5$	$t = 0.6$	$t = 0.7$	$t = 0.8$	$t = 0.9$
0.1	1.7701E-4	2.5821E-4	2.7527E-4	2.8030E-4	2.8209E-4	2.8281E-4	2.8313E-4	2.8328E-4	2.8335E-4
0.2	1.7476E-4	2.5594E-4	2.7284E-4	2.7781E-4	2.7958E-4	2.8030E-4	2.8062E-4	2.8076E-4	2.8084E-4
0.3	1.7328E-4	2.5369E-4	2.7042E-4	2.7535E-4	2.7710E-4	2.7781E-4	2.7812E-4	2.7827E-4	2.7834E-4
0.4	1.7181E-4	2.5146E-4	2.6802E-4	2.7290E-4	2.7464E-4	2.7534E-4	2.7565E-4	2.7579E-4	2.7587E-4
0.5	1.7035E-4	2.4924E-4	2.6565E-4	2.7047E-4	2.7219E-4	2.7289E-4	2.7319E-4	2.7334E-4	2.7341E-4
0.6	1.6890E-4	2.4704E-4	2.6329E-4	2.6806E-4	2.6977E-4	2.7046E-4	2.7076E-4	2.7090E-4	2.7097E-4
0.7	1.6746E-4	2.4486E-4	2.6094E-4	2.6568E-4	2.6736E-4	2.6804E-4	2.6834E-4	2.6848E-4	2.6856E-4
0.8	1.6603E-4	2.4269E-4	2.5862E-4	2.6331E-4	2.6497E-4	2.6565E-4	2.6595E-4	2.6609E-4	2.6616E-4
0.9	1.6460E-4	2.4055E-4	2.5632E-4	2.6096E-4	2.6261E-4	2.6327E-4	2.6357E-4	2.6371E-4	2.6378E-4
1.0	1.6319E-4	2.3842E-4	2.5403E-4	2.5862E-4	2.6026E-4	2.6092E-4	2.6121E-4	2.6135E-4	2.6142E-4

Consider the time-fractional KdV–Burgers equation [15–19] as follows

$$D_t^{\alpha} u + \varepsilon u u_x - \nu u_{xx} + \mu u_{xxx} = 0 \tag{1.2}$$

where ε, ν and μ are constants and α $(0 < \alpha \le 1)$ represents the order of fractional derivative. In this numerical technique, the fractional derivative has been discretized by Grünwald–Letnikov derivative and hence the fractional KdVB equation transformed to a finite difference equation, that has been adjusted in the form of implicit finite difference scheme.

Numerous numerical and analytical methods have been initiated in recent past in order to analyse the classical KdV–Burgers equation. Methods such as the decomposition method [20], tanh method [21], hyperbolic tangent method and exponential rational function approach [22], Septic B-spline method [23], Radial basis functions [24], Quartic B-spline Galerkin approach [25], modified Bernstein polynomial [26] and quintic B-spline finite elements [27] had been developed independently and had been used to acquire numerical as well as exact solutions of KdVB equation. But, till now no numerical work has been reported to find the solution of fractional KdVB equation. Methods such as Adomian decomposition method [28] and homotopy perturbation method [15] were utilised to acquire the approximate solution of fractional KdVB equation. The explicit and approximate solutions of the nonlinear fractional KdVB equation were presented in [17].

This manuscript emphasizes on the implementation of Petrov–Galerkin technique for solution of time-fractional KdVB equation with a perception to manifest the abilities of this present technique in dealing with nonlinear equation. The primary intention is to demonstrate the competence and reliability of Petrov–Galerkin technique in solving time-fractional KdV–Burgers equation.

Table 3
L_2 and L_∞ error norms for KdV–Burgers equation using Petrov–Galerkin method at various points of t taking $\alpha = 1$, $\varepsilon = 6$, $\nu = 0.0005$ and $\mu = 0.1$.

t	L_2	L_∞
0.1	1.29814E-10	8.90530E-11
0.2	3.24729E-10	2.22708E-10
0.3	4.55738E-10	3.12239E-10
0.4	6.51429E-10	4.46194E-10
0.5	7.83639E-10	5.36188E-10
0.6	9.80090E-10	6.70419E-10
0.7	1.11351E-9	7.60861E-10
0.8	1.31070E-9	8.95346E-10
0.9	1.44534E-9	9.86221E-10
1.0	1.61436E-9	9.16034E-10

2. Fractional derivative and integrals

There are several approaches to define the derivatives of fractional order such as Grünwald–Letnikov, Riemann–Liouville and Caputo. Riemann–Liouville fractional derivative is not suitable for real world physical problems as it requires the definition of fractional order initial conditions, which have no physically meaningful explanation. Caputo introduced an alternative definition, which has the advantage of defining integer order initial conditions for fractional order differential equations.

Definition. The Grünwald–Letnikov fractional derivative of a function $f(t)$, is defined as [29–31]

$${}_aD_t^{\alpha} f(t) = \lim_{\substack{h \to 0 \\ nh = t-a}} h^{-\alpha} \sum_{r=0}^{n} \omega_r^{\alpha} f(t - rh) \tag{2.1}$$

where $\omega_r^{\alpha} = (-1)^r \binom{\alpha}{r}$,

$\omega_0^{\alpha} = 1$ and $\omega_r^{\alpha} = \left(1 - \frac{\alpha+1}{r}\right)\omega_{r-1}^{\alpha}, \quad r = 1, 2, \ldots$