Chaos, Solitons and Fractals 116 (2018) 414–423

Contents lists available at ScienceDirect

Chaos, Solitons and Fractals

Nonlinear Science, and Nonequilibrium and Complex Phenomena

journal homepage: www.elsevier.com/locate/chaos

Derivation of a groundwater flow model within leaky and self-similar aquifers: Beyond Hantush model

Ramotsho Amanda, Abdon Atangana*

Institute of Groundwater studies, Faculty of Natural and Agricultural Science, University of the Free State, 9300 Bloemfontein, South Africa

ARTICLE INFO

Article history:
Received 1 September 2018
Revised 13 September 2018
Accepted 13 September 2018

Keywords:
Leaky aquifers with fractals setting
Fractal derivative and self-similarity
Adam Bashforth (AB) method
New two-step Laplace (AB) method

ABSTRACT

Scientists conducted researches on leaky aquifers, developed groundwater models and equations for this type of aquifer. These equations are developed from classical formulas like Darcy's Law. The classical equations cannot be used to assess heterogeneous and porous aquifers. A fractal differential operator is then defined for fractal geometry and it is now widely used by many scientists to allow better understanding of such aquifers, which cannot be based on Euclid geometry or assessed using classical formulas. Fractals have a property called self-similarity and there are examples in nature like fractured aquifers. A self-similar leaky aquifer is visualized as a heterogeneous media, which cannot be solved or assessed using the classical equation. This article entails a new groundwater equation derived for flow within a self-similar leaky aquifer. We provided in this paper new models of groundwater flowing within a leaky aquifers with self-similarities. The new model is more representative that one suggested by Hantush as it includes in the mathematical formulation the scaling effect of a geological formation. Thus the solution is not only function of time-space and the well-known aquifer's parameters including the transmissivity, storativity and leaky factor, but there is a new factor that takes into account the scaling of the aquifer. We presented the existence and uniqueness of the solutions. We used a newly established numerical scheme to solve the new equations. We presented some numerical simulations and observed very interesting features that are observed in real world problems.

1. Introduction

Confined and unconfined aquifers can be classified as leaky aquifers, only if the underlying and the overlying aquitards are leaking and recharging the main aquifer [1]. Groundwater model can be of a good use if its mathematical equations are close approximates of the physical objects been described. Many studies have been conducted to understand groundwater flow using derivation of subsurface flow equations and applying mathematical models. The Theis equation is one of the fundamental solutions for the deterministic mathematical models of groundwater flow. A model proposed by Theis was modified using the fractional order derivatives, which have limits. Hantush added water leakage as a volumetric sink/source term in the groundwater equation to make use of the equation when investigating a leaky aquifer [2].

The derived groundwater flow equation for flow within a leaky aquifer is derived using classical equations like Darcy's Law, which are based on Euclid geometry. The equation cannot be used to investigate heterogeneous aquifers. However, a mathematician Mandelbrot firstly invented fractals in order to relate the objects with mathematics. Fractals can be found in nature patterns and are the new approach used in mathematics and art researches to simplify and explain concepts that cannot be solved in terms of Euclid geometry. A fractal is a never-ending pattern that repeats itself at different scales.

Fractals have fine structure at arbitrarily small scales, they are too irregular to be easily described in traditional Euclidean geometry, they have a dimension, which is non-integer, and they have a simple and recursive definition [1,2]. Fractals are made up of the most three important properties namely self-similarity, scaling and statistics. The main focus of this article is on the self-similarity property. Self-similarity occurs when a fractal replicate itself and shows iteration throughout [2].

Imagine a fractured system that replicate itself in a scaled version but with the same characteristics. As we know groundwater flow modeling is based on the Darcy's Law, which is valid when the flow is laminar. Fractured rocks often have two-flow regimes fast flow in the fractures and slow flow in rock matrix with long residence time. Darcy's Law can describe the flow in the matrix,

* Corresponding author.
E-mail addresses: ramotshoa@dws.gov.za (R. Amanda), atanganaa@ufs.ac.za (A. Atangana).

https://doi.org/10.1016/j.chaos.2018.09.025

but it is not valid in the fractures or conduits where the flow may be tubular [3].

2. Fractal derivative

Fractal derivative is defined as a nonstandard type of derivative in which the variable t has been scaled according to t^{α}. The well-known differentiation equation in calculus is given as:

$$f'(a) = \lim_{h\to 0} \frac{f(a+h) - f(a)}{h} \tag{1}$$

Wen Chen introduced a Hausdorff derivative, which was compared with the fractal derivative for anomalous diffusion. Hausdorff derivative is local in nature and the fractal derivative is non local in nature where function $g(t)$ was introduced with respect to a fractal measure t^{α}. The symbol α only appears once on the Hausdorff equation than o the fractional derivative and they both take into account the scale space-time transforms. Atangana and Goufo used a new derivative called the local variable order derivative to further enhance the model of groundwater flow in a leaky aquifer and they incorporated Iterative method where steps are repeated into the mathematical formula, which includes the complexity of the leaky aquifer [4].

Atangana and Bildik converted the Thiem and the Theis groundwater flow equation to the time-fractional groundwater flow model by using fractional order derivative to predict the groundwater flow. Atangana and Goufo realized that the modified equation by Atangana does not accommodate the complexity of the aquifer and proposed a new version that can accommodate leaky aquifers also [5]. Atangana, Djida and Area developed an application of new numerical scheme of Atangana–Baleanu fractional integral to groundwater flow within a leaky aquifer. They described and analyzed the groundwater model using differentiation with non-local and non-singular kernel [6].

2.1. Derivation of a new groundwater flow equation for flow within leaky aquifer through advection and dispersion

Water can flow within an aquifer in two ways, through advection and dispersion. The water through a porous media doesn't travel at an average speed and also the direction changes. The groundwater storage equation is given below.

$$\frac{\partial v}{\partial t} = Q_1 + Q_3 - Q_2 \tag{2}$$

Where Q_3 is the leakage into the aquifer recharging the main aquifer, Q_1 is the water flowing into the aquifer and Q_2 is the total water been discharged or out of the aquifer and $\frac{\partial v}{\partial t}$ represents the storage of the aquifer.

$$\frac{\partial v}{\partial t} = S(2\pi r)dr\frac{\partial h}{\partial t} \tag{3}$$

$$Q_1 = 2\pi(r+\Delta r)b\left[\frac{\partial}{\partial r}\left(\frac{\partial h}{\partial r}\right)\Delta r + \frac{\partial h}{\partial r}\right]K \tag{4}$$

$$Q_2 = 2\pi rb\left[\frac{\partial h}{\partial r}\right]K \tag{5}$$

$$Q_3 = 2\pi r\Delta r\frac{h(r,t)}{\lambda^2}Kb \tag{6}$$

$$S(2\pi r)dr\frac{\partial h}{\partial t} = 2\pi(r+dr)b\left[\frac{\partial}{\partial r}\left(\frac{\partial h}{\partial r}\right)\Delta r + \frac{\partial h}{\partial r}\right]K + 2\pi r\Delta r\frac{h(r,t)}{\lambda^2}Kb - 2\pi rb\left[\frac{\partial h}{\partial r}\right]K \tag{7}$$

$$\frac{S(2\pi r)\Delta r\frac{\partial h}{\partial t}}{2\pi r\Delta r} = \frac{2\pi(r+\Delta r)b\left[\frac{\partial}{\partial r}\left(\frac{\partial h}{\partial r}\right)\Delta r + \frac{\partial h}{\partial r}\right]K}{2\pi r\Delta r} + \frac{2\pi rdr\frac{h(r,t)}{\lambda^2}Kb}{2\pi r\Delta r} - \frac{2\pi rb\left[\frac{\partial h}{\partial r}\right]K}{2\pi r\Delta r} \tag{8}$$

$$S\frac{\partial h}{\partial t} = \frac{(r+\Delta r)b\left[\frac{\partial}{r}\left(\frac{\partial h}{\partial r}\right)\Delta r + \frac{\partial h}{\partial r}\right]K}{r\Delta r} + \frac{h(r,t)}{\lambda^2}Kb - \frac{b\left[\frac{\partial h}{\partial r}\right]K}{\Delta r} \tag{9}$$

Geo-hydrological equations used to determine parameters can be used to better understand and simply equations. We know that transmissivity equals to conductivity and thickness as given in literature.

$$T = Kb \tag{10}$$

$$S\frac{\partial h}{\partial t} = T\frac{(r+\Delta r)\left[\frac{\partial}{\partial r}\left(\frac{\partial h}{\partial r}\right)\Delta r + \frac{\partial h}{\partial r}\right]}{r\Delta r} + T\frac{h(r,t)}{\lambda^2} - T\frac{\left[\frac{\partial h}{\partial r}\right]}{\Delta r} \tag{11}$$

Transmissivity can be taken into the other side by dividing both sides with transmissivity and getting storativity over transmissivity and $\frac{(r+\Delta r)}{r\Delta r}$, can be simplified to $(\frac{1}{\Delta r} + \frac{1}{r})$:

$$\frac{S}{T}\frac{\partial h}{\partial t} = \frac{(r+\Delta r)\left[\frac{\partial}{\partial r}\left(\frac{\partial h}{\partial r}\right)\Delta r + \frac{\partial h}{\partial r}\right]}{r\Delta r} + \frac{h(r,t)}{\lambda^2} - \frac{\left[\frac{\partial h}{\partial r}\right]}{\Delta r} \tag{12}$$

$$\frac{S}{T}\frac{\partial h}{\partial t} = \left(\frac{1}{\Delta r} + \frac{1}{r}\right)\left[\frac{\partial}{\partial r}\left(\frac{\partial h}{\partial r}\right)\Delta r + \frac{\partial h}{\partial r}\right] + \frac{h(r,t)}{\lambda^2} - \frac{\left[\frac{\partial h}{\partial r}\right]}{\Delta r} \tag{13}$$

Factorization is used in mathematics to help in breaking brackets and for further simplification of the equation. Applying factorization in the equation above results in the following equation and like terms will be cancelled after factorizing:

$$\frac{S}{T}\frac{\partial h}{\partial t} = \frac{\partial}{\partial r}\left(\frac{\partial h}{\partial r}\right) + \frac{1}{r}\frac{\partial}{\partial r}\left(\frac{\partial h}{\partial r}\right)\Delta r + \frac{1}{\Delta r}\frac{\partial h}{\partial r} + \frac{1}{r}\frac{\partial h}{\partial r} + \frac{h(r,t)}{\lambda^2} - \frac{1}{\Delta r}\left[\frac{\partial h}{\partial r}\right] \tag{14}$$

$$\frac{S}{T}\frac{\partial h}{\partial t} = \frac{\partial}{\partial r}\left(\frac{\partial h}{\partial r}\right) + \frac{1}{r}\frac{\partial}{\partial r}\left(\frac{\partial h}{\partial r}\right)dr + \frac{1}{r}\frac{\partial h}{\partial r} + \frac{h(r,t)}{\lambda^2} \tag{15}$$

$$\frac{S}{T}\frac{\partial h}{\partial t} = \frac{\partial}{\partial r}\left(\frac{\partial h}{\partial r}\right)\left(1 + \frac{\Delta r}{r}\right) + \frac{1}{r}\frac{\partial h}{\partial r} + \frac{h(r,t)}{\lambda^2} \tag{16}$$

The above Eq. (15) is more representative that one suggested by Hantush as it includes in the mathematical formulation the scaling effect of a geological formation. Thus the solution is not only function of time-space and the well-known aquifer's parameters including the transmissivity, storativity and leaky factor, but there is a new factor that takes into account the scaling of the aquifer.

2.2. Flow within a self-similar leaky aquifer

Flow and storage of water within a fractured rock aquifer can be very complex and complicated due to the heterogeneity associated with the mechanical discontinuity resulting from the presence of fractures [[7–12]]. Fractal derivative is introduced in the water flowing into the aquifer and the water flowing out of the aquifer. The change will only be implemented into Q_1 and Q_2 only.

$$\frac{\partial v}{\partial t} = {Q_1}^{\alpha} + {Q_2}^{\alpha} + Q_3 \tag{16}$$

$${Q_1}^{\alpha} = 2\pi(r+\Delta r)b\left[\frac{\partial}{\partial r^{\alpha}}\left(\frac{\partial h}{\partial r^{\alpha}}\right)\Delta r + \frac{\partial h}{\partial r^{\alpha}}\right]K \tag{17}$$