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On purely nonlinear oscillators generalizing an isotonic potential

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ABSTRACT

(1) We consider a nonlinear generalization of the isotonic oscillator with an asymmetric potential. (2) Using a symmetrization principle we construct a symmetric potential. (3) The period function in this potential has the same value as in the original asymmetric potential. (4) It is amplitude dependent and expressible in terms of the hypergeometric function.

1. Introduction

In recent times there has been a considerable amount of interest in purely nonlinear oscillators for which the restoring elastic force is proportional to $sgn(x)|x|^{\alpha}$, with x representing the displacement and α being any positive real number [1–5]. The presence of the signum function ensures that the force is an odd function for all values of α. The potential $x^{4/3}$ was examined in detail in [6]. In previous studies of one-dimensional conservative oscillatory systems it was customary to assume that the restoring force involves odd integer powers of the displacement.

The reason for being interested in such purely nonlinear oscillators is because of the potential for their applications in diverse areas of science and engineering. For instance it is known that the stress–strain properties of several materials used in aircraft manufacturing, ceramic industries, composites, polyurethane foam etc are strongly nonlinear and the usual polynomial approximations to the restoring force is generally inadequate. Secondly the nonlinearity of the restoring force is often due not to the physical properties of materials but to geometrical consequences of the system such as its shape, loading etc. For example helicoidal and conical springs made of materials having linear properties arise due to their geometry and thereby cause nonlinearity of the restoring force. An area where non-integer order nonlinearity is of particular significance is in the design of micro-electro-mechanical systems (MEMS), nano-electro-mechanical devices, vibration, acoustic and impact isolators. Mechanical microstructures such as sensors, valves, gears etc are particularly important in nanotechnology; and it is plausible that the observed differences between the results of simulation and actual measurements in experimental devices is most likely due to errors in modeling caused by the assumption of integer nonlinearity [7] (Chapter 2 and references therein).

From a theoretical point of view it is always desirable to have exact or accurate analytical approximations for the solutions of such equations. Lyapunov showed that when the restoring force is proportional to an odd integer power of the displacement then the solutions can be expressed in terms of the Jacobi elliptic functions cn and sn respectively.

On the other hand for systems of the form

$$\ddot{x} + c_{\alpha}^{2} sgn(x)|x|^{\alpha} = 0$$

or the allied system

$$\ddot{x} + c_{\alpha}^{2} x|x|^{\alpha-1} = 0$$

where α is not necessarily a positive integer the solutions may be described by Ateb functions [8] which are the inverses of the incomplete Beta function. The term Ateb was coined by Rosenberg (Beta read backwards). Senik [9,10] showed that the Ateb functions are actually the solutions of the differential equations

$$\dot{x} = y^{\alpha}, \qquad \dot{y} = -\frac{2}{\alpha+1}x,$$

namely $x(t) = sa(1,\alpha,t)$ and $y(t) = ca(\alpha,1,t)$ and that these may be expressed in terms of the three-argument ca and sa functions. The

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inverse of the incomplete Beta function is defined by

$$B(a,b)=\int_0^{0\leq t\leq 1} z^{a-1}(1-z)^{b-1}dz,$$

and it may be verified that the inverse of the half of the incomplete Beta function $\frac{1}{2}B(\frac{1}{2},\frac{1}{\alpha+1})$ coincides with $x(t)$ on $[-\frac{1}{2}\Pi_\alpha,\frac{1}{2}\Pi_\alpha]$ where $\Pi_\alpha := B(\frac{1}{(\alpha+1)},\frac{1}{2})$ denotes the usual Beta function. Furthermore it is known that $sa(\alpha,1,t)$ and $ca(1,\alpha,t)$ are odd and even functions of $t\in\mathbb{R}$ having period $2\Pi_\alpha$. They satisfy the identity $sa^2(\alpha,1,t)+ca^{\alpha+1}(1,\alpha,t)=1$ and are referred to as the sine Ateb and cosine Ateb functions respectively [11,12]. In addition their first derivatives are given by

$$\frac{d}{dt}ca(\alpha,1,t)=-\frac{2}{\alpha+1}sa(1,\alpha,t),\quad \frac{d}{dt}sa(1,\alpha,t)=ca^{\alpha}(\alpha,1,t).$$

The above identities clearly demonstrate their similarity with the trigonometric sine and cosine functions. In fact just as the trigonometric functions yield the normal mode vibrations of a linear system the class of Ateb functions are solutions of normal mode vibrations of certain nonlinear multicomponent systems [8]. The approximations of Ateb functions by smooth elementary functions have been considered in [13,14].

As a generalization of the potential $x^{4/3}$ an exact formula for the time period of oscillation of the system

$$\ddot{x}+c_\alpha^2 sgn(x)|x|^\alpha=0$$

subject to the initial conditions $x(0)=A$ and $\dot{x}(0)=0$ is given by [12],

$$T=\sqrt{\frac{8\pi}{c_\alpha^2(\alpha+1)}}\frac{\Gamma\left(\frac{1}{\alpha+1}\right)}{\Gamma\left(\frac{\alpha+3}{2(\alpha+1)}\right)}|A|^{(1-\alpha)/2},$$

with Γ representing the Euler gamma function. This formula reduces to $2\pi/c_\alpha$ when $\alpha=1$ which corresponds to the linear harmonic oscillator. The solution may be expressed by the three-argument Ateb ca function

$$x=Aca(\alpha,1,\omega_{ca}t),$$

with frequency given by

$$\omega_{ca}=|A|^{(1-\alpha)/2}\sqrt{\frac{c_\alpha^2(\alpha+1)}{2}}.$$

It is evident that the frequency is in general amplitude dependent, unless $\alpha=1$, which corresponds to the linear situation.

In this article we use a symmetrization procedure due to Mañosas and Torres [15] to derive the time period of purely nonlinear oscillators. Using their arguments we reproduce the result for the time period of the system

$$\ddot{q}+c_\alpha^2 sgn(q)|q|^\alpha=0 \qquad \text{where} \qquad c_\alpha>0,$$

obtained earlier by Cveticanin [4,5,7,11]. We then consider a generalization of the standard isotonic potential and propose a purely nonlinear generalized isotonic system in a spirit similar to the generalization of the linear harmonic oscillator stated above. The potential of a linear harmonic oscillator (LHO) given by, $\omega^2x^2/2$, is a rational function having a minimum at the origin $x=0$ and is symmetric. The LHO is characterized by the fact that its time period is independent of the amplitude. Although there are several instances of differential systems exhibiting periodic motion, it is indeed rare to find systems displaying periodic motion with an amplitude independent time period. Such systems are said to be isochronous and apart from the LHO there is only one isochronous system with a rational potential namely the isotonic oscillator [16]. Its equation of motion given by, $\ddot{x}+\omega^2x=\ell^2/x^3$, is nonlinear and admits the general solution

$$x(t)=(Ax_1^2+2Bx_1x_2+Cx_2^2)^{1/2},$$

where $x_1(t)$ and $x_2(t)$ are any linearly independent solutions to the equation of the LHO having Wronskian ℓ^2 and where A, B and C satisfy the condition $AC-B^2=\ell^2$. The equation may be derived from the potential, $V(x)=\omega^2x^2/2+\ell^2/2x^2$, which consists of two branches separated by the asymptote $x=0$ with each branch being an asymmetric curve displaying a minima. Physically a system governed by the isotonic potential corresponds to the simplest two-body case of the N-body translational invariant Calogero model [17] and is of great interest in quantum optics [18] and in the theory of coherent states [19,20]. Our main result for the proposed generalized isotonic oscillator may be stated as follows:

Theorem 1.1. *For the purely nonlinear generalized isotonic oscillator governed by a potential*

$$U(q)=\frac{c_\alpha}{8}\left(|q|^{\frac{\alpha+1}{2}}-\frac{1}{|q|^{\frac{\alpha+1}{2}}}\right)^2,$$

with α being a positive real number, with equation of motion given by

$$\ddot{q}+\frac{c_\alpha}{8}(\alpha+1)q^\alpha=\frac{c_\alpha(\alpha+1)}{8q^{\alpha+2}},\quad 0<q<\infty.$$

subject to initial conditions $q(0)=q_0$ and $\dot{q}(0)=0$, the time period, T, is given by

$$T=\frac{4}{\sqrt{c_\alpha}(\alpha+1)}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\binom{\frac{2}{\alpha+1}}{2m}\binom{\frac{1}{\alpha+1}-\frac{1}{2}-m}{n}k_\alpha^{2(m+n)}B(m+n+1/2,1/2).$$

The **organization** of the paper is as follows. In Section 2 we derive the result for the time period of the purely nonlinear oscillator introduced above, not only for the sake of completeness but to also outline the general strategy which will be adopted to deal with potentials which are not necessarily symmetric about the origin. This is followed by a brief discussion of the standard isotonic oscillator which is an example of an isochronous system in section 3 for which the potential is asymmetric. By exploiting the results contained in [15] we show how an equivalent (as far as the period function is concerned) symmetric potential may be constructed in such a situation. This is followed in Section 4 by an analysis of the period function of a generalized isotonic oscillator which involves non-integer nonlinear dependence.

2. Time period of, $\ddot{q}+c_\alpha^2 sgn(q)|q|^\alpha=0$, using symmetrization argument

Consider the equation

$$\ddot{q}+c_\alpha^2 sgn(q)|q|^\alpha=0, \tag{2.1}$$

with initial conditions $q(0)=q_0$ and $\dot{q}(0)=0$ and $c_\alpha>0$. This c_α can be normalized to unity by rescaling the time. The expression for the potential energy is given by

$$U(q)=\frac{c_\alpha^2}{\alpha+1}|q|^{(\alpha+1)}$$

For every q there exists $\sigma(q)$ such that $U(\sigma(q))=U(q)$ with $q\sigma(q)<0$. We define the function g by

$$g(q)=sgn(q)\sqrt{U(q)}=sgn(q)\frac{c_\alpha}{\sqrt{\alpha+1}}|q|^{(\alpha+1)/2}. \tag{2.2}$$

It is obvious that $g(0)=0$ and $g'(0)>0$. Moreover one can easily deduce that

$$g^{-1}(q)=sgn(q)\left(\frac{\sqrt{\alpha+1}}{c_\alpha}|q|\right)^{2/(\alpha+1)} \tag{2.3}$$

We also observe that

$$g(\sigma(q))=sgn(\sigma(q))\sqrt{U(\sigma(q))}=\frac{sgn(\sigma(q))}{sgn(q)}g(q)=-g(q)$$

so that

$$\sigma(q)=g^{-1}(-g(q)).$$

This provides that relation for the explicit determination of $\sigma(q)$ given q [15]. It now follows that in the present situation we have $\sigma(q)=-q$

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