ARTICLE IN PRESS

International Journal of Non-Linear Mechanics ▮ (▮▮▮▮) ▮▮▮–▮▮▮

Contents lists available at ScienceDirect

International Journal of Non-Linear Mechanics

journal homepage: www.elsevier.com/locate/nlm

Exact solutions of the models of nonlinear diffusion in an inhomogeneous medium

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ARTICLE INFO

Keywords:
Nonlinear diffusion
Inhomogeneous media
Invariant submodels
Invariant solutions
Nonlinear integral equations
"black holes"

ABSTRACT

The main purpose of our research is to find as many as possible the solutions of the equation of the general model of nonlinear diffusion in an inhomogeneous medium and to establish their physical meaning and to apply the obtained solutions to the description nonlinear diffusion processes in an inhomogeneous medium. To achieve of this purpose the basic submodels (possessing nontrivial symmetry properties) of the general model are obtained and researched. The formulas of the production of the new solutions for the equations of these submodels are obtained. For these submodels all invariant submodels are found. The essentially distinct invariant solutions (not connected with a help of the point transformations) describing these invariant submodels are found either explicitly, or their search is reduced to the solving of the nonlinear integral equations. The physical meanings of these solutions are established. Some of the 25 explicitly found solutions describe a diffusion process only for a finite period of time, others — for an infinite period of time. Some solutions describe a nonlinear diffusion process either with fixed or evolving "black holes", in the vicinity of which the concentration infinitely increases. The presence of the arbitrary constants in the integral equations, that determine other 27 solutions opens up the new opportunities for analytical and numerical study of the boundary value problems for the received submodels, and, thus, for the original model of the nonlinear diffusion process. For such invariant submodels, we are studied the diffusion processes, for which at the initial instant of the time at a fixed point either a concentration and rate of its change, or concentration and its gradient are given. The solving of the boundary value problems describing these processes reduces to the solving of nonlinear integral equations. The existence and uniqueness of the solutions of these boundary value problems under certain conditions are established. A mechanical relevance of the obtained solutions is as follows: 1) these solutions describe specific nonlinear diffusion processes in an inhomogeneous medium, 2) these solutions can be used as test solutions in the numerical calculations, which perform in the studies of the real diffusion processes, 3) these solutions make it possible to assess the degree of adequacy of a given mathematical models to the real physical processes, after carrying out experiments corresponding to these solutions, and estimating the resulting deviations.

The obtained results can be used to study the diffusion of substances, the diffusion of conduction electrons and other particles, the diffusion of physical fields, the propagation of heat in an inhomogeneous medium.

1. Introduction

Many mathematical models of physics and continuum mechanics are formulated in the form of linear and quasi-linear differential equations. The model is a representation (scheme) phenomena more simple than the original, but it reflects the basic properties of this phenomena. Mathematical model is a description of the real scheme by mathematical language. In the derivation of these equations, the mechanics and physics used an invariance of the phenomenon under transformations, as a consequence of the symmetries of space–time, which describe the phenomenon. The set of transformations acts in the space–time around us that allows to represent the geometric structure of the space. But symmetries may be hidden, they may be the result of physical properties of the phenomena being modeled. The use of symmetry properties allow correctly simulate the phenomena and to classify the submodels. The symmetry analysis of the equations of the models of physics and mechanics of continuous media is one of the most effective ways to obtain quantitative and qualitative characteristics of the physical processes. The role of transformation groups in the construction and study of mathematical models on the example of the important mechanical theories (nonlinear elasticity and fluids of grade n) was studied in [1]. The modern concept of the symmetry analysis is understood as the fullest using of the group of transformations admitted by the equations

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https://doi.org/10.1016/j.ijnonlinmec.2018.08.004
Received 15 April 2018; Received in revised form 6 August 2018; Accepted 7 August 2018
Available online xxxx

of model primarily to obtain and research the exact solutions. Exact solutions allow us to describe the specific physical processes. Exact solutions can be used as test solutions in numerical calculations, which perform in the studies of the real processes. Exact solutions allow us to assess the degree of adequacy of a given mathematical model to the real physical processes, after carrying out experiments corresponding to these decisions, and estimating the deviations that arise.

Diffusion (from Latin "diffusion" - spreading) is the movement of a fluid particles, leading to the transfer of substances and uniform concentration or to the equilibrium distribution of particle concentrations in the medium. The diffusion phenomenon occurs not only for the substances, but also for conduction electrons and other particles and for the physical fields.

In this paper we study the general model of nonlinear diffusion in an inhomogeneous medium. This model is described by the equation:

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left(\varphi(x)\,\psi(u)\,\frac{\partial u}{\partial x}\right), \tag{1}$$

where $u = u(t,x)$ is a substance concentration at the point $x \in (-\infty,\infty)$ at the time t; $D = \varphi(x)\,\psi(u)$ is a diffusion coefficient characterizing a diffusion rate; $\varphi(x)$ and $\psi(u)$ are any smooth function that satisfy to the condition

$$\varphi'(x)\,\psi'(u) \neq 0. \tag{2}$$

This condition means that diffusion is nonlinear and medium is inhomogeneous.

The main purpose of our research is to find as many as possible the solutions of the equation of the general model of nonlinear diffusion in an inhomogeneous medium and to establish their physical meaning and to apply the obtained solutions to the description nonlinear diffusion processes in an inhomogeneous medium.

Notes. 1. *Eq. (1) with the condition (2), also describes the nonlinear process of the heat propagation in an inhomogeneous rod. In this case, $u = u(t,x)$ is the temperature at the point x of the rod at the time t.*

2. *All results obtained for the Eq. (1), due to the transformations $t = \int a(\tau)\,d\tau,\ x = \int \frac{1}{b(y)}\,dy$, are carried over to the diffusion equation with the nonstationary diffusion coefficient*

$$\frac{\partial u}{\partial \tau} = a(\tau)\,b(y)\,\frac{\partial}{\partial y}\left(f(y)\,g(u)\,\frac{\partial u}{\partial y}\right),$$

Symmetry properties and the simplest invariant solutions of Eq. (1) for some particular values of the functions $\varphi(x)$ and $\psi(u)$ were studied in many papers (see for example, [2–12]). We will carry out complete classification of all invariant solutions of this equation for all the functions $\varphi(x)$ and $\psi(u)$ and indicate the physical meaning of these solutions.

2. Basic submodels

We study the group properties of Eq. (1).

2.1. Group classification

The Eq. (1) is a first-order conservation law. This allows us to introduce an additional function $w = w(t,x)$, such that Eq. (1) is written in the form of an equivalent system of the first order:

$$\frac{\partial u}{\partial x} = \frac{w}{\varphi(x)\,\psi(u)}, \quad \frac{\partial w}{\partial x} = \frac{\partial u}{\partial t}. \tag{3}$$

We will fulfill group classification of the system (3). We will solve the problem of the group classification of this system using the algorithm proposed in [13,14]. In contrast to the classical algorithm presented in [15], this algorithm, firstly, avoids the considerable analytical difficulties associated with the analysis of the classifying equations that arise when applying the algorithm from [15]; second, it substantially reduces the number of calculations. This algorithm has been successfully used in [10,11,16–21] for group classification of the various equations of mechanics and mathematical physics.

An arbitrary element of this system is $\boldsymbol{f} = (\varphi(x), \psi(u))$. Structure equations of an arbitrary element are written as follows:

$$\boldsymbol{f}_t = 0, \quad \boldsymbol{f}_w = 0, \quad \varphi_u = 0, \quad \psi_x = 0. \tag{4}$$

The operator of generalized equivalence transformations of the system (3) is defined as

$$\xi^0(t,x,u,w)\,\partial_t + \xi^1(t,x,u,w)\,\partial_x + \eta^1(t,x,u,w)\,\partial_u + \eta^2(t,x,u,w)\,\partial_w$$
$$+\zeta(t,x,u,w,\boldsymbol{f})\cdot\partial_{\boldsymbol{f}},$$

where $\xi^0, \xi^1, \eta^1, \eta^2, \zeta$ are smooth functions of their variables.

The condition of invariance of the manifold determined by Eqs. (3), (4) to this operator, with allowance [13,14] for the rule of extension of this operator after splitting in terms of parametric derivatives yields a system of the equations determining the generalized equivalence transformations of the system (3) and the specializations of the arbitrary element $\boldsymbol{f}$.

Solutions of this overdetermined system are all specializations of the arbitrary element and the corresponding equivalence transformations of the system (3). These equivalence transformations form the set of generalized equivalence transformations of the system (3). For the system (3) the set of the generalized equivalence transformations of this system coincides with the group of its universal equivalence transformations. For the specializations of the arbitrary element we study the action of the group of equivalences of the system (3) with this arbitrary element or, more exactly, the action of the factor-group of this group of equivalences by the kernel of the main groups of the system (3) on the system (3) with this arbitrary element. As a result of this action, equivalent systems are formed. To find all non-equivalent systems, we construct an optimal system of subgroups for the considered group of equivalences or, more exactly, for the factor-group of this group of equivalences by the kernel of the main groups of the system (3). The equivalence transformations acting on $\boldsymbol{f}$ identically form the kernel of the main groups of the system (3) with this arbitrary element $\boldsymbol{f}$, i.e., they are admitted by the system (3) for all elements $\boldsymbol{f}$ possessing the considered arbitrariness. In addition to the kernel of the main groups, the system (3) admits each subgroup of the group of equivalences under the condition that this subgroup acts on the element $\boldsymbol{f}$ identically. For each subgroup of the constructed optimal system of subgroups, the element $\boldsymbol{f}$ is specified under the condition that this subgroup acts on the element $\boldsymbol{f}$ identically. The final results of the group classification of the system (3) under condition (2) we formulate for the physically significant Eq. (1).

• The kernel of the main groups of Eq. (1) is generated by the operator

$$X_0 = \partial_t.$$

• For

$$\varphi = x^\alpha, \left(\frac{1}{(\ln\psi)'}\right)'' \neq 0, \quad \alpha = \text{const} \neq 0 \tag{5}$$

the main group of Eq. (1) is generated by the operator X_0 and operator

$$X_1(\alpha) = (2-\alpha)\,t\partial_t + x\partial_x$$

• For

$$\varphi = \exp x, \quad \left(\frac{1}{(\ln\psi)'}\right)'' \neq 0 \tag{6}$$

the main group of Eq. (1) is generated by the operator X_0 and operator

$$X_2 = t\partial_t - \partial_x$$

• For

$$\left(\frac{1}{(\ln\varphi)'}\right)''(\ln\varphi)'' \neq 0, \quad \psi = u^\beta, \quad \beta = \text{const} \neq 0 \tag{7}$$

the main group of Eq. (1) is generated by the operator X_0 and operator

$$X_3(\beta) = -\beta t\partial_t + u\partial_u$$

Please cite this article in press as: Yu.A. Chirkunov, Exact solutions of the models of nonlinear diffusion in an inhomogeneous medium, International Journal of Non-Linear Mechanics (2018), https://doi.org/10.1016/j.ijnonlinmec.2018.08.004.