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Continuum theory for elastic sheets formed by inextensible crossed elasticae

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ABSTRACT

An equilibrium theory for elastic surfaces formed by two families of intextensible fibers is developed. This extends the work of Wang and Pipkin, which accounts for the intrinsic flexural stiffness of the fibers, to include intrinsic twisting resistance. The constituent fibers behave mechanically like spatial Kirchhoff rods. These are regarded as being continuously distributed on the surface and convected by the underlying surface deformation as inextensible material curves.

1. Introduction

Wang and Pipkin [1,2] developed a theory of fiber-reinforced elastic surfaces that accounts for the flexural stiffness of the fibers at the constitutive level. This extends a model for perfectly flexible fiber-reinforced sheets introduced by Rivlin ([3]; see also Chapter 7 of [4]) and further developed by Pipkin [5–7]. The extended model fits into the framework of the second-gradient theory of elasticity [8–13]. Our purpose in the present work is to enhance the Wang–Pipkin model by including a constitutive sensitivity to fiber twist. In the Kirchhoff theory of rods [14], twist is kinematically independent of the deformation of the curve describing the trajectory of the rod. It may be treated in the framework of the Cosserat theory of elasticity [15,16] in which a rotation field accounts for the rate of change of the orientation of a rod cross section – the twist – with respect to arclength along the rod. This is partially coupled to the underlying deformation by requiring the unit tangent to the curve describing the rod trajectory to be convected as a material vector.

In the present work we assume the elasticae comprising the surface to be pinned at their points of intersection in such a way as to allow them to pivot freely about the evolving unit normal to the deforming sheet. This constraint effectively means that the surface normal is embedded in the fiber cross sections. Accordingly, fiber twist is ultimately determined by the deformation of the underlying surface. In this way the relative tractability of strain-gradient elasticity is preserved while accounting for the physically important effects of fiber twisting resistance. A model of this kind, without inextensibility constraints, was introduced in [17] to describe the mechanics of woven fabrics. Associated numerical solutions exhibiting unusual internal bending transition layers – corroborating a prediction made in [2] – are described in [18,19], and some theoretical aspects of the model are discussed in [20]. However, fiber inextensibility introduces a number of unusual mathematical features that justify an independent treatment.

In Section 2 we recount the kinematical foundations of the model in terms of the differential geometry of the fiber network, drawing on results derived in [17,21] as needed. The basic constitutive framework is developed in Section 3. There we introduce an areal strain-energy function modeled after Kirchhoff's rod theory. This is used to extract the relevant response functions from a virtual-work statement. These in turn are used, in Section 4, to obtain the equilibrium equations via a variational argument. In Section 5, we use global force and moment balances to aid in the interpretation of the various terms arising in the theory. Comparison with the Rivlin–Pipkin theory for perfectly flexible fibers is briefly discussed in Section 6, and finally, in Section 7, we modify an example discussed in [1] to highlight the essential role played by fiber twist in the buckling response of a rectangular sheet.

In contrast to [1], we adopt the general curvilinear coordinate formalism of standard shell theory [22], to elucidate the underlying tensorial character of the model.

2. Surface geometry and deformation

2.1. Surface geometry

We use convected coordinates θ^α; $\alpha = 1, 2$, to label material points of the lattice, regarded as a two-dimensional manifold. The function $\mathbf{x}(\theta^\alpha)$ furnishes position of a material point on a fixed reference plane Ω. Position of the same material point on the deformed surface ω is denoted by $\mathbf{r}(\theta^\alpha)$. The latter parametrization induces the associated tangent-basis elements $\mathbf{a}_\alpha = \mathbf{r}_{,\alpha}$; the metric $a_{\alpha\beta} = \mathbf{a}_\alpha \cdot \mathbf{a}_\beta$; the dual metric $(a^{\alpha\beta}) = (a_{\alpha\beta})^{-1}$ and the dual tangent basis $\mathbf{a}^\alpha = a^{\alpha\beta}\mathbf{a}_\beta$. These in turn yield the local orientation of ω in terms of its unit normal $\mathbf{n}$, defined by $\mathbf{n} = \mathbf{a}_1 \times \mathbf{a}_2 / |\mathbf{a}_1 \times \mathbf{a}_2|$. Here and henceforth a comma followed by a Greek subscript is used to identify a partial derivative with respect to the corresponding coordinate.

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The Gauss and Weingarten equations of surface theory are

$$\mathbf{r}_{,\alpha\beta} = \Gamma^{\lambda}_{\alpha\beta}\mathbf{a}_{\lambda} + b_{\alpha\beta}\mathbf{n} \quad \text{and} \quad \mathbf{n}_{,\alpha} = -b_{\alpha\beta}\mathbf{a}^{\beta}, \tag{1}$$

where $\Gamma^{\lambda}_{\alpha\beta}$ are the Levi-Civita connection coefficients – symmetric with respect to interchange of the subscripts – induced by the coordinates on ω, and $b_{\alpha\beta}$ is the symmetric covariant curvature tensor (the coefficients of the second fundamental form).

The lattice is assumed to consist of two families of fibers that are continuously distributed over the surface; thus, every material point lies at the intersection of a pair of fibers, each modeled as a mathematical curve. The unit tangents to the fibers on ω are denoted by $\mathbf{l}$ and $\mathbf{m}$, and their counterparts on Ω by $\mathbf{L}$ and $\mathbf{M}$. These admit the representations

$$\mathbf{l} = l^{\alpha}\mathbf{a}_{\alpha}, \quad \mathbf{m} = m^{\alpha}\mathbf{a}_{\alpha} \quad \text{and} \quad \mathbf{L} = L^{\alpha}\mathbf{e}_{\alpha}, \quad \mathbf{M} = M^{\alpha}\mathbf{e}_{\alpha}, \tag{2}$$

in terms of contravariant components, for example. Here $\mathbf{e}_{\alpha}$ are the basis vectors induced by the coordinates on Ω via $\mathbf{e}_{\alpha} = \mathbf{x}_{,\alpha}$, and $\mathbf{e}^{\alpha}$ are their duals. The fibers are presumed to be convected as material curves, implying [21] that

$$\lambda l^{\alpha} = L^{\alpha} \quad \text{and} \quad \mu m^{\alpha} = M^{\alpha} \tag{3}$$

relating contravariant components only, where λ and μ are the fiber stretches. The fiber shear angle γ on ω is defined by

$$\sin\gamma = \mathbf{l}\cdot\mathbf{m}. \tag{4}$$

We assume, as in [1], that the fibers form a uniform rectangular grid on Ω and that they remain inextensible in the course of deformation; i.e., that $\lambda = \mu = 1$. A number of general formulas developed in [17,21] are needed in the present work. Their specializations to initially orthogonal, inextensible fibers are recalled here. For example, the natural tangent basis is [21]

$$\mathbf{a}_{\alpha} = L_{\alpha}\mathbf{l} + M_{\alpha}\mathbf{m}, \tag{5}$$

the areal stretch is

$$J = |\cos\gamma|, \tag{6}$$

and the covariant surface curvature is

$$b_{\alpha\beta} = \kappa_l L_{\alpha}L_{\beta} + \kappa_m M_{\alpha}M_{\beta} + \tau(L_{\alpha}M_{\beta} + M_{\alpha}L_{\beta}), \tag{7}$$

where

$$\kappa_l = b_{\alpha\beta}l^{\alpha}l^{\beta} \quad \text{and} \quad \kappa_m = b_{\alpha\beta}m^{\alpha}m^{\beta} \tag{8}$$

are the normal curvatures of the deformed fibers, and

$$\tau = b_{\alpha\beta}l^{\alpha}m^{\beta} \tag{9}$$

is the torsion. This is not the conventional surface twist. The latter is the off-diagonal term of the curvature matrix relative to an orthonormal tangent basis on ω. We elaborate on this distinction in the next subsection.

The geodesic curvatures η_l and η_m of the fibers are defined by [21]

$$l^{\alpha}\mathbf{l}_{,\alpha} = \eta_l\mathbf{p} + \kappa_l\mathbf{n} \quad \text{and} \quad m^{\alpha}\mathbf{m}_{,\alpha} = \eta_m\mathbf{q} + \kappa_m\mathbf{n}, \tag{10}$$

with

$$\mathbf{p} = \mathbf{n}\times\mathbf{l} \quad \text{and} \quad \mathbf{q} = \mathbf{n}\times\mathbf{m}. \tag{11}$$

We also have [21]

$$m^{\alpha}\mathbf{l}_{,\alpha} = \phi_l\mathbf{p} + \tau\mathbf{n} \quad \text{and} \quad l^{\alpha}\mathbf{m}_{,\alpha} = \phi_m\mathbf{q} + \tau\mathbf{n}, \tag{12}$$

where ϕ_l and ϕ_m are the so-called Tchebychev curvatures of the fibers.

It is well known that the geodesic curvatures are determined by the surface metric. In the present specialization to fiber inextensibility and a uniform rectangular lattice of fibers on Ω, the explicit expressions are [17,21]

$$J\eta_l = L^{\alpha}(\sin\gamma)_{,\alpha} \quad \text{and} \quad J\eta_m = -M^{\alpha}(\sin\gamma)_{,\alpha}. \tag{13}$$

Our further work is facilitated by introducing

$$\mathbf{r}_{|\alpha\beta} = \mathbf{r}_{,\alpha\beta} - \bar{\Gamma}^{\lambda}_{\alpha\beta}\mathbf{r}_{,\lambda}, \tag{14}$$

where $\bar{\Gamma}^{\lambda}_{\alpha\beta}$ are the connection coefficients on Ω and $(\cdot)_{|\alpha\beta}$ is the second covariant derivative with respect to the metric of Ω. The Gauss equation $(1)_1$ then furnishes

$$\mathbf{r}_{|\alpha\beta} = S^{\lambda}_{\alpha\beta}\mathbf{r}_{,\lambda} + b_{\alpha\beta}\mathbf{n}, \tag{15}$$

where

$$S^{\lambda}_{\alpha\beta} = \Gamma^{\lambda}_{\alpha\beta} - \bar{\Gamma}^{\lambda}_{\alpha\beta}. \tag{16}$$

In [17] it is shown that this admits the fiber decomposition

$$\mathbf{r}_{|\alpha\beta} = L_{\alpha}L_{\beta}(\mathbf{g}_l + \kappa_l\mathbf{n}) + M_{\alpha}M_{\beta}(\mathbf{g}_m + \kappa_m\mathbf{n}) + (L_{\alpha}M_{\beta} + M_{\alpha}L_{\beta})(\boldsymbol{\Gamma} + \tau\mathbf{n}), \tag{17}$$

where

$$\mathbf{g}_l = \eta_l\mathbf{p}, \quad \mathbf{g}_m = \eta_m\mathbf{q} \tag{18}$$

and

$$\boldsymbol{\Gamma} = \phi_l\mathbf{p} = \phi_m\mathbf{q}. \tag{19}$$

The Tchebychev curvatures appearing in these expressions may be determined by exploiting the symmetry of $S^{\lambda}_{\alpha\beta}$ with respect to interchange of the subscripts. In [17] it is demonstrated that this leads to

$$J\phi_l = J\eta_m + M^{\alpha}(\sin\gamma)_{,\alpha} \quad \text{and} \quad J\phi_m = J\eta_l - L^{\alpha}(\sin\gamma)_{,\alpha}, \tag{20}$$

and hence, with (13), to the conclusion that the curvatures vanish identically: $\phi_l = \phi_m = 0$. Thus,

$$\boldsymbol{\Gamma} = \mathbf{0}, \tag{21}$$

identically. These conclusions do not follow in the case of fiber extensibility.

From (17) and (19) we infer that

$$\tau\mathbf{n} = L^{\alpha}M^{\beta}\mathbf{r}_{|\alpha\beta}. \tag{22}$$

If the deformed surface is a plane ($b_{\alpha\beta} = 0$), then τ vanishes in particular and this reduces to $\mathbf{r}_{uv} = \mathbf{0}$, where u and v are rectangular coordinates aligned with the initial fiber trajectories. In this case we have Rivlin's representation [3]

$$\mathbf{r} = \mathbf{f}(u) + \mathbf{g}(v) \tag{23}$$

of the deformation. This representation has been used in [23,24] to analyze complex plane deformations of inextensible lattices.

2.2. Fiber kinematics

Consider the orthonormal basis $\{\mathbf{l}_i\} = \{\mathbf{l}, \mathbf{p}, \mathbf{n}\}$ with $\mathbf{p}$ given by $(11)_1$. This consists of the unit tangent $\mathbf{l}$ to the first fiber trajectory, and two vectors – $\mathbf{p}$ and the surface normal $\mathbf{n}$ – spanning the cross-sectional plane of the fiber. Let $\{\mathbf{L}_i\} = \{\mathbf{L}, \mathbf{M}, \mathbf{N}\}$, where $\mathbf{N}$ is the unit normal to Ω and $\mathbf{M} = \mathbf{N}\times\mathbf{L}$. Then there is a rotation tensor, $\mathbf{R}_{(l)}$, such that $\mathbf{l}_i = \mathbf{R}_{(l)}\mathbf{L}_i$. The rate of change of the basis $\{\mathbf{l}_i\}$ with respect to arclength along the $\mathbf{L}$-trajectory, denoted by $(\cdot)'$, is

$$\mathbf{l}_i' = L^{\alpha}\mathbf{l}_{i,\alpha} = \boldsymbol{\omega}_{(l)}\times\mathbf{l}_i, \tag{24}$$

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