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Spherical analysis on homogeneous vector bundles ☆

Fulvio Ricci *, Amit Samanta

Scuola Normale Superiore, Piazza dei Cavalieri 7, 56126 Pisa, Italy

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ABSTRACT

Given a Lie group G, a compact subgroup K and a representation $\tau \in \widehat{K}$, we assume that the algebra of $\mathrm{End}(V_\tau)$-valued, bi-τ-equivariant, integrable functions on G is commutative. We present the basic facts of the related spherical analysis, putting particular emphasis on the rôle of the algebra of G-invariant differential operators on the homogeneous bundle E_τ over G/K. In particular, we observe that, under the above assumptions, (G, K) is a Gelfand pair and show that the Gelfand spectrum for the triple (G, K, τ) admits homeomorphic embeddings in some $\mathbb{C}^n$.

In the second part, we develop in greater detail the spherical analysis for $G = K \ltimes H$ with H nilpotent. In particular, for $H = \mathbb{R}^n$ and $K \subset SO(n)$ and for H equal to the Heisenberg group H_n and $K \subset U(n)$, we characterize the representations $\tau \in \widehat{K}$ giving a commutative algebra.

0. Introduction

Let (G, K) be a Gelfand pair with G a Lie group and K a compact subgroup of it. Recent work has put attention on the fact that the spherical analysis on the bi-K-invariant

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* Corresponding author.
E-mail addresses: fricci@sns.it (F. Ricci), amit.gablu@gmail.com (A. Samanta).

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algebra $L^1(K\backslash G/K)$ gains new interesting aspects from the fact that its Gelfand spectrum Σ, i.e., the space of bounded spherical functions with the compact-open topology, can be naturally embedded into some Euclidean space as a closed set [8].

Such an embedding ρ is defined by choosing a generating k-tuple $(D_1, \dots, D_k)$ in the algebra of G-invariant differential operators on G/K and assigning to the spherical function ϕ on G/K the vector $\rho(\phi) = \big(\lambda_{D_1}(\phi), \dots, \lambda_{D_k}(\phi)\big) \in \mathbb{C}^k$ whose entries $\lambda_{D_j}(\phi)$ are the eigenvalues of ϕ under the D_j's.

This allows to introduce a notion of smoothness for functions defined on Σ, and to pose the problem, classical in Fourier analysis, of relating smoothness of the spherical transform of a given bi-K-invariant function on G with properties of the function itself. This question has been investigated in detail for *nilpotent Gelfand pairs*, in which $G = K \ltimes H$ is a motion group on a nilpotent group H [9–11].

In this paper we extend the basic framework for such analysis to the spherical transform of type τ, where τ is an irreducible unitary representation of K for which the appropriate commutativity assumptions are satisfied.

The notion of spherical transform of type τ goes back to [15]. In most of the existing literature the accent is on the case where (G, K) is a symmetric pair. For the general case we refer to [28, Ch. 6] and [25,3].

There are at least two equivalent ways to introduce the objects of our analysis on a triple (G, K, τ), where G is a Lie group, K a compact subgroup of it and $\tau \in \widehat{K}$ as above.

In the first (matrix-valued) picture one considers the homogeneous bundle $E_\tau = G \times_\tau V_\tau$ with basis G/K and linear operators on sections of E_τ commuting with the action of G. The Schwartz kernel theorem implies that, under mild continuity assumptions, any such operator can be represented by convolution with an $\mathrm{End}(V_\tau)$-valued[1] distributional kernel F on G satisfying the identity

$$F(k_1 x k_2) = \tau(k_2^{-1}) F(x) \tau(k_1^{-1}). \tag{0.1}$$

The commutativity condition imposed on τ is that the algebra of $\mathrm{End}(V_\tau)$-valued integrable functions F on G satisfying the above identity is commutative with respect to the convolution defined in (1.4) below.

In the second (scalar-valued) picture one considers the algebra of integrable scalar-valued functions f on G which are K-central and satisfy the identity

$$f * \overline{\chi}_\tau = f,$$

and the requirement on τ is that this algebra be commutative.

We say that (G, K, τ) is a *commutative triple* if either of these conditions is satisfied.

[1] For V, W finite dimensional complex vector spaces, $\mathrm{Hom}(V, W)$ denotes the space of linear operators from V to W. If $V = W$, we write $\mathrm{End}(V)$ instead of $\mathrm{Hom}(V, V)$.