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Real hyperbolic hyperplane complements in the complex hyperbolic plane

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This paper studies Riemannian manifolds of the form $M \setminus S$, where M^4 is a complete four dimensional Riemannian manifold with finite volume whose metric is modeled on the complex hyperbolic plane $\mathbb{CH}^2$, and S is a compact totally geodesic codimension two submanifold whose induced Riemannian metric is modeled on the real hyperbolic plane $\mathbb{H}^2$. In this paper we write the metric on $\mathbb{CH}^2$ in polar coordinates about S, compute formulas for the components of the curvature tensor in terms of arbitrary warping functions (Theorem 7.1), and prove that there exist warping functions that yield a complete finite volume Riemannian metric on $M \setminus S$ whose sectional curvature is bounded above by a negative constant (Theorem 1.1(1)). The cases of $M \setminus S$ modeled on $\mathbb{H}^n \setminus \mathbb{H}^{n-2}$ and $\mathbb{CH}^n \setminus \mathbb{CH}^{n-1}$ were studied by Belegradek in [4] and [3], respectively. One may consider this work as "part 3" to this sequence of papers.

1. Introduction

Let M be a complete (connected) locally symmetric Riemannian manifold with finite volume and negative sectional curvature, and let S be a (possibly disconnected) compact

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totally geodesic codimension two submanifold of M. It is known that the pair (M, S) is modeled on $(\mathbb{H}^n, \mathbb{H}^{n-2})$, $(\mathbb{CH}^n, \mathbb{CH}^{n-1})$, or the "exceptional case" $(\mathbb{CH}^2, \mathbb{H}^2)$. In [4] and [3] Belegradek provides an in depth study of $M \setminus S$, the manifold obtained from M by "drilling out" S, when the model for the pair (M, S) is one of the first two situations. Here we consider the exceptional case, when (M, S) is modeled on $(\mathbb{CH}^2, \mathbb{H}^2)$.

The main result proved in this paper is the following.

Theorem 1.1. *If M is a complete finite volume complex hyperbolic 2-manifold and S is a compact totally real totally geodesic 2-dimensional submanifold, then*

1. *$M \setminus S$ admits a complete finite volume metric with sectional curvature ≤ -1.*
2. *$M \setminus S$ admits a complete finite volume A-regular metric with sectional curvature < 0.*

It should be noted that such pairs (M, S) modeled on $(\mathbb{CH}^2, \mathbb{H}^2)$ are known to exist, but constructing new families of such pairs is difficult and an open area of active research. One way to construct such pairs is to quotient $\mathbb{CH}^2$ by arithmetic lattices of "simple type" (see Section 4 of [6] and the corresponding references). Another method is to construct $\mathbb{R}$-Fuchsian representations of surface groups into $PU(2, 1)$. For more information here, please see [20] and the references therein.

The manifold $M \setminus S$ is diffeomorphic to a compact manifold N obtained by cutting out a tubular neighborhood of S in M and removing all cusps in the case that M is not compact. There are two possible types of boundary components of N. The first is compact infranil manifolds, which arise as cross-sections of the cusps of M (if any), and the second type is circle bundles over the components of S.

All three statements in the following Corollary 1.2 can be deduced from other known results, as will be discussed in Remark 1.3. But combining Theorem 1.1 with the methods deployed in [4] and [3] provides independent proofs of these statements.

Corollary 1.2. *Suppose that M is a complete finite volume complex hyperbolic 2-manifold and S is a compact totally real totally geodesic 2-dimensional submanifold. Then*

1. *the group $\pi_1(M \setminus S)$ is non-elementary (strongly) relatively hyperbolic, where the peripheral subgroups are the fundamental groups of the ends of $M \setminus S$.*
2. *$\pi_1(M \setminus S)$ satisfies the Farrell–Jones isomorphism conjecture.*
3. *$\pi_1(M \setminus S)$ satisfies the Rapid Decay Property and the Baum–Connes conjecture.*

Remark 1.3. The proof that Corollary 1.2(1) follows from Theorem 1.1(1) is identical to its analogues in [3] and [4]. In particular, see Section 12 of [3] in conjunction with Theorem 4.2 in [4]. Many other properties of $\pi_1(M \setminus S)$ are then known to follow from Corollary 1.2(1) (see most conclusions of Theorem 1.1 in [4] and Theorem 1.4 in [3]). But the fact that $\pi_1(M \setminus S)$ is relatively hyperbolic relative to the fundamental groups