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The transfer is functorial

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We prove that the Becker–Gottlieb transfer is functorial up to homotopy, for all fibrations with finitely dominated fibers. This resolves a lingering foundational question about the transfer, which was originally defined in the late 1970s in order to simplify the proof of the Adams conjecture. Our approach differs from previous attempts in that we closely emulate the geometric argument in the case of a smooth fiber bundle. This leads to a "multiplicative" description of the transfer, different from the standard presentation as the trace of a diagonal map.

Contents

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1. Introduction

A transfer is a construction that takes a map $X \to Y$ to some "wrong way" map from Y to X. The prototype of a transfer was probably first defined by Schur in 1902 as a construction in group theory [20]. A half a century later, the transfer idea was imported into the field of algebraic topology as a wrong way map in cohomology.

In 1972, Roush defined a transfer for finite covering spaces $X \to Y$ as a stable wrong way map $\Sigma^\infty Y_+ \to \Sigma^\infty X_+$ [19]. In 1975, this idea was expanded on by Becker and Gottlieb to define transfers for smooth fiber bundles with closed manifold fiber [2]. This was used to give a "remarkably simple proof"[1] of the Adams conjecture. In 1976, Becker and Gottlieb produced a purely homotopy theoretic construction of the transfer using fiberwise S-duality, thereby extending the definition to the class of Hurewicz fibrations with finitely dominated fibers [3]. For a history of such constructions and their applications, see [4].

The transfer is most elementary to describe for a smooth fiber bundle $p: E \to B$ with closed manifold fibers and compact base B. We first observe that p admits a fiberwise smooth embedding $E \subset B \times \mathbb{R}^n$ for n suitably large. Let τ and ν be the vertical tangent and vertical normal bundles of p, respectively. The Pontryagin–Thom construction gives a map $\Sigma^n(B_+) \to E^\nu$, where E^ν denotes the Thom space of ν. Composing this with the inclusion of the zero section of the tangent bundle, we get a map of spaces

$$\Sigma^n(B_+) \to E^\nu \subset E^{\nu\oplus\tau} = \Sigma^n(E_+)\,.$$

This defines a stable map $p^!: \Sigma^\infty B_+ \to \Sigma^\infty E_+$, which is the transfer. It is well-defined up to contractible choice as n tends to ∞.

As the Pontryagin–Thom construction is functorial on compositions of embeddings, it is straightforward to deduce that for any composition of smooth fiber bundles with closed manifold fibers

$$X \xrightarrow{p} Y \xrightarrow{q} Z$$

the transfer is functorial up to homotopy, $(q \circ p)^! \simeq p^! \circ q^!$ (cf. §3).

Given that the transfer is defined for a larger class of fibrations, it is reasonable to ask how much more generally functoriality holds. Previous work by other authors gives functoriality for fiber bundles with a compact Lie group G as the structure group and finite G-CW complex fibers [14, th. IV.7.1], and functoriality for "parametrized Euclidean

[1] In the words of Peter May, see the review of [2], MR0377873.