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On the mean field type bubbling solutions for Chern–Simons–Higgs equation

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A B S T R A C T

This paper is the second part of our comprehensive study on the structure of the solutions for the following Chern–Simons–Higgs equation:

$$\begin{cases} \Delta u + \frac{1}{\varepsilon^2} e^u(1-e^u) = 4\pi \sum_{j=1}^N \delta_{p_j}, & \text{in } \Omega, \\ u \ \text{ is doubly periodic on } \partial\Omega, \end{cases} \tag{0.1}$$

where Ω is a parallelogram in $\mathbb{R}^2$ and $\varepsilon > 0$ is a small parameter. In part 1 [29], we proved the non-coexistence of different bubbles in the bubbling solutions and obtained an existence result for the Chern–Simons type bubbling solutions under some nearly necessary conditions. Mean field type bubbling solutions for (0.1) have been constructed in [27]. In this paper, we shall study two other important issues for the mean field type bubbling solutions: the necessary conditions for the existence and the local uniqueness. The results in this paper lay the foundation to find the exact number of solutions for (0.1).

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1. Introduction

This is the second paper of our research on the structure of the solutions for the following Chern–Simons–Higgs equation:

$$\begin{cases} \Delta u + \frac{1}{\varepsilon^2} e^u (1 - e^u) = 4\pi \sum_{j=1}^N \delta_{p_j} & \text{in } \Omega, \\ u \text{ is doubly periodic on } \partial\Omega, \end{cases} \tag{1.1}$$

where $\delta_p(x)$ is the Dirac measure at $p \in \Omega$, $\varepsilon > 0$ is the Chern–Simons constant, and Ω is a flat torus in $\mathbb{R}^2$.

Equation (1.1) was derived from gauge theory proposed by Jackiw–Weinberg [20] and Hong–Kim–Pac [18] to explain physics of higher temperature superconductivity. After the works [18,20], various Chern–Simons theories with non-abelian gauge fields have been introduced to various applications in physics. See [1,2,8], [12–17] [19,21,22], [27–35], [38,39]. These Chern–Simons systems, after a suitable ansatz, can be reduced to systems of nonlinear elliptic partial differential equations, which have posed many mathematically challenging problems. In order to understand those difficult problems for the Chern–Simons systems, it is fundamentally important to completely understand the solution structure of (1.1) as $\varepsilon \to 0$.

To eliminate the Dirac measure in the right hand side of (1.1), we introduce the Green function $G(x, p)$ of $-\Delta$ in Ω with singularity at p, subject to the doubly periodic boundary condition. That is, $G(x, p)$ satisfies

$$\begin{cases} -\Delta G(x,p) = \delta_p - \frac{1}{|\Omega|}, & \int_\Omega G(x,p)\, dx = 0, \\ G(x,p) \text{ is doubly periodic on } \partial\Omega, \end{cases}$$

where $|\Omega|$ is the measure of Ω. Let

$$u_0(x) = -4\pi \sum_{j=1}^N G(x, p_j). \tag{1.2}$$

Using this function u_0, (1.1) can be re-written as

$$\begin{cases} \Delta u + \frac{1}{\varepsilon^2} e^{u+u_0}(1 - e^{u+u_0}) = \frac{4N\pi}{|\Omega|}, & \text{in } \Omega, \\ u \text{ is doubly periodic on } \partial\Omega. \end{cases} \tag{1.3}$$

In this paper, we will study (1.1), or equivalently (1.3) for small $\varepsilon > 0$. In [9], Choe and Kim initiated the study of the asymptotic behavior for the solutions of (1.3). In particular, they proved the following result.

Theorem A. *For any sequence of solution u_n of (1.3) with $\varepsilon_n > 0$, one of the followings holds true:*