## Full Length Article

# On weighted approximation with Jacobi weights 

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#### Abstract

We obtain matching direct and inverse theorems for the degree of weighted $L_{p}$-approximation by polynomials with the Jacobi weights $(1-x)^{\alpha}(1+x)^{\beta}$. Combined, the estimates yield a constructive characterization of various smoothness classes of functions via the degree of their approximation by algebraic polynomials. In addition, we prove Whitney type inequalities which are of independent interest. (c) 2018 Elsevier Inc. All rights reserved.


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## 1. Introduction and main results

In this paper, we are interested in weighted polynomial approximation with the Jacobi weights

$$
w_{\alpha, \beta}(x):=(1-x)^{\alpha}(1+x)^{\beta}, \quad \alpha, \beta \in J_{p}:= \begin{cases}(-1 / p, \infty), & \text { if } 0<p<\infty \\ {[0, \infty),} & \text { if } p=\infty\end{cases}
$$

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Let $L_{p}^{\alpha, \beta}(I):=\left\{f \mid\left\|w_{\alpha, \beta} f\right\|_{L_{p}(I)}<\infty\right\}$, where $\|\cdot\|_{L_{p}(I)}$ is the usual $L_{p}$ (quasi)norm on the interval $I \subseteq[-1,1]$, and, for $f \in L_{p}^{\alpha, \beta}(I)$, denote by

$$
E_{n}(f, I)_{\alpha, \beta, p}:=\inf _{p_{n} \in \mathbb{P}_{n}}\left\|w_{\alpha, \beta}\left(f-p_{n}\right)\right\|_{L_{p}(I)}
$$

the error of best weighted approximation of $f$ by polynomials in $\mathbb{P}_{n}$, the set of algebraic polynomials of degree not more than $n-1$. For $I=[-1,1]$, we denote $\|\cdot\|_{p}:=\|\cdot\|_{L_{p}[-1,1]}$, $L_{p}^{\alpha, \beta}:=L_{p}^{\alpha, \beta}([-1,1]), E_{n}(f)_{\alpha, \beta, p}:=E_{n}(f,[-1,1])_{\alpha, \beta, p}$, etc.

Definition 1.1 ([10]). For $r \in \mathbb{N}_{0}$ and $0<p \leq \infty$, denote $\mathbb{B}_{p}^{0}\left(w_{\alpha, \beta}\right):=L_{p}^{\alpha, \beta}$ and

$$
\mathbb{B}_{p}^{r}\left(w_{\alpha, \beta}\right):=\left\{f \mid f^{(r-1)} \in A C_{l o c}(-1,1) \quad \text { and } \quad \varphi^{r} f^{(r)} \in L_{p}^{\alpha, \beta}\right\}, \quad r \geq 1
$$

where $\varphi(x):=\sqrt{1-x^{2}}$ and $A C_{l o c}(-1,1)$ denotes the set of functions which are locally absolutely continuous in $(-1,1)$.

We remark that, in the case $p<1$, our definition of derivatives is understood in the classical sense, i.e., the assumption $f^{(r-1)} \in A C_{l o c}(-1,1)$ in the case $r \geq 2$ is understood in the sense that $f$ is the $(r-1)$ st integral of a locally absolutely continuous $\bar{f}^{(r-1)}$ plus a polynomial of degree $r-2$.

As is common when dealing with $L_{p}$ spaces, we will not distinguish between a function in $\mathbb{B}_{p}^{r}\left(w_{\alpha, \beta}\right)$ and all functions which are equivalent to it in $L_{p}^{\alpha, \beta}$.

Definition 1.2 ([10]). For $k, r \in \mathbb{N}$ and $f \in \mathbb{B}_{p}^{r}\left(w_{\alpha, \beta}\right), 0<p \leq \infty$, define

$$
\begin{equation*}
\omega_{k, r}^{\varphi}\left(f^{(r)}, t\right)_{\alpha, \beta, p}:=\sup _{0 \leq h \leq t}\left\|\mathcal{W}_{k h}^{r / 2+\alpha, r / 2+\beta}(\cdot) \Delta_{h \varphi(\cdot)}^{k}\left(f^{(r)}, \cdot\right)\right\|_{p} \tag{1.1}
\end{equation*}
$$

where

$$
\mathcal{W}_{\delta}^{\xi, \zeta}(x):=(1-x-\delta \varphi(x) / 2)^{\xi}(1+x-\delta \varphi(x) / 2)^{\zeta}
$$

and

$$
\Delta_{h}^{k}(f, x):= \begin{cases}\sum_{i=0}^{k}\binom{k}{i}(-1)^{k-i} f\left(x-\frac{k h}{2}+i h\right), & \text { if }\left[x-\frac{k h}{2}, x+\frac{k h}{2}\right] \subseteq[-1,1] \\ 0, & \text { otherwise },\end{cases}
$$

is the $k$ th symmetric difference.
For $\delta>0$, denote (see [9])

$$
\mathfrak{D}_{\delta}:=\{x|1-\delta \varphi(x) / 2 \geq|x|\} \backslash\{ \pm 1\}=[-1+\mu(\delta), 1-\mu(\delta)]
$$

where

$$
\mu(\delta):=2 \delta^{2} /\left(4+\delta^{2}\right)
$$

We note that $\mathfrak{D}_{\delta_{1}} \subset \mathfrak{D}_{\delta_{2}}$ if $\delta_{2}<\delta_{1} \leq 2$, and that $\mathfrak{D}_{\delta}=\emptyset$ if $\delta>2$. Also, since $\Delta_{h \varphi(x)}^{k}(f, x)=0$ if $x \notin \mathfrak{D}_{k h}$,

$$
\begin{equation*}
\omega_{k, r}^{\varphi}\left(f^{(r)}, t\right)_{\alpha, \beta, p}=\sup _{0<h \leq t}\left\|\mathcal{W}_{k h}^{r / 2+\alpha, r / 2+\beta}(\cdot) \Delta_{h \varphi(\cdot)}^{k}\left(f^{(r)}, \cdot\right)\right\|_{L_{p}\left(\mathfrak{D}_{k h}\right)} \tag{1.2}
\end{equation*}
$$

In particular, $\omega_{k, r}^{\varphi}\left(f^{(r)}, t\right)_{\alpha, \beta, p}=\omega_{k, r}^{\varphi}\left(f^{(r)}, 2 / k\right)_{\alpha, \beta, p}$, for all $t \geq 2 / k$.
Following [10] we also define the weighted averaged moduli.

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