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Discrete and continuous green energy on compact manifolds

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Abstract

In this note we study the role of the Green function for the Laplacian in a compact Riemannian manifold as a tool for obtaining well-distributed points. In particular, we prove that a sequence of minimizers for the Green energy is asymptotically uniformly distributed. We pay special attention to the case of locally harmonic manifolds.

MSC: 31C12

Keywords: Green energy; Well-distributed points; Harmonic manifold

1. Introduction

Distributing points in spheres or other sets is a very classical problem. Its modern formulation in terms of energy-minimizing configurations is due to the discoverer of the electron J. J. Thomson who in 1904 posed the question (rephrased here) *in which position – within some set such as a ball or a sphere – would N electrons lie in order to minimize their electrostatic potential?* [50].

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Thomson's question was related to a certain atomic model – the plum pudding model – which had a very short life due to the spectacular advances of experimental physics in the beginning of the XX century. The question still remained as a beautiful problem to be solved, and gained importance for different applications in the subsequent years. In 1930, the botanist Tammes suggested that the (astonishingly regular) distribution of pores in pollen particles followed a pattern that maximized the minimal distance between pores (see [49] for Tammes' original publication and [30] for high definition images). This idea gave an excellent explanation to the fact that there are barely pollen particles with 5 or 11 pores (since if 5 pores can be placed in the surface of a sphere then 6 equal sized pores can also be placed, and similarly for 11 and 12. The mathematical proof of this fact was not complete until the 1980s, see [25,43,12,21]). See [53,33] for two classical reviews on the problem.

A seminal paper [41] launched a new collection of works on the topic of distributing points in spheres. The problem had gained new motivation with Shub and Smale's approach to polynomial system solving, which in the one-dimensional case required to find a polynomial all of whose zeros were well-conditioned in a particular sense. In [44] they proved that such zeros correspond (via the stereographic projection) to points in the Riemann sphere which maximize the product of their mutual distances, equivalently, points with minimal logarithmic energy. This relation led Smale to include the problem of algorithmically finding these points in his list of problems for the XXI Century [46]. See [5,38] for recent surveys on Smale's problem.

There are many different approaches to the definition of what a *sensibly distributed* collection of spherical points is. Apart from the mentioned minimization of the energy and maximization of minimal distances, other definitions include having small discrepancy, providing exact integral formulas for low degree polynomials (spherical t-designs, see [11,24] for a recent breakthrough), having optimal covering radius, maximizing the sum of the mutual distances, etc. There are dozens of papers on each of these problems. A very recent and very complete survey on the problem is [17].

In a recent paper [6], the problem of minimizing the logarithmic energy in the 2-sphere was rewritten as a facility location problem: that of allocating a number of heat sources in such a way that the average temperature is maximized. This approach led to some nontrivial results including upper bounds for the logarithmic energy of well-separated sequences with small discrepancy. As a consequence it was proved that a sequence of minimizers of Riesz's s-energy is asymptotically minimizing for the logarithmic energy (the reciprocal of this fact was proved in Leopardi's paper [34]).

The logarithmic energy is defined by

$$E_{\log}(x_1, \dots, x_N) = \sum_{i \neq j} \log \|x_i - x_j\|^{-1}, \quad x_i \in \mathbb{S}^2.$$

This function has a very special property: its (spherical) Laplacian is constant. This follows from the fact that the function $\log \|x - y\|^{-1}$ is (up to scaling) the Green function for the Laplacian in $\mathbb{S}^2$. A collection of points minimizing the logarithmic energy is called a set of *elliptic Fekete points*, though sometimes the word "elliptic" is omitted. See [26,51,29] for an introduction to the classical theory.

The Riesz s-energy is defined by

$$E_s(x_1, \dots, x_N) = \sum_{i \neq j} \|x_i - x_j\|^{-s}.$$

Remarkable progress in the study of logarithmic and Riesz energies (minimum values, properties of the minimal energy configurations, relation to separation distance, spherical cap discrepancy and cubature formulas...) has taken place in the last three decades, see for example