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Dynamical response of a Timoshenko beams on periodical nonlinear supports subjected to moving forces

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ABSTRACT

The periodically supported Timoshenko beam subjected to moving forces has been investigated by numerous researches. The existed models have been developed for linear supports, and this article presents a new one for nonlinear supports. By using a periodic condition and the Fourier series development, the dynamic equation of the Timoshenko beam leads to a relation between the beam displacements and the reaction forces of the supports. This relation does not depend on the support behaviour and it exists also for the Euler-Bernoulli beam. Then, the responses can be obtained by combining this relation and the constitutive law of the supports. A numerical method based on discretization of the time and frequency responses has been developed for nonlinear supports. Moreover, the influence of the beam model has been studied with numerical examples of linear and nonlinear supports. The results show that the Timoshenko beam should be used for the moving forces with high speed and/or the supports with large stiffness.

1. Introduction

The periodically supported beam subjected to moving forces has been investigated in numerous publications [1–10]. In these articles, the Euler-Bernoulli or Timoshenko beams resting on identical supports at periodical intervals have been considered in steady-state. The response to moving forces are calculated analytically when the supports are linear. However, these models cannot be extended easily for nonlinear supports. Recently, the dynamics of a periodically supported beam has been represented by the system equivalence [11] by using a periodic condition of reaction forces. This model could work for nonlinear supports, but the author has not presented a method to compute the dynamical responses. Some other researches have considered the model of beams on nonlinear foundations (i.e. the beam is supported continuously) by using the perturbation technique [12–14], the Galerkin method [15] or the numerical methods [16,17].

This article presents a complete analytic model for the dynamics of beams resting on periodic nonlinear supports. A relation between the beam displacement and the reactions forces has been established from the periodic condition and the dynamic equation of the Timoshenko beam. Then, a numerical method has been developed to compute the response form this relation and the constitutive law of the nonlinear supports. Moreover, a comparison between the Timoshenko and Euler-Bernoulli beam models has been performed with numerical examples of linear and nonlinear behaviours.

2. Periodically supported Timoshenko beam

2.1. Dynamical equations in steady-state

Let's consider an infinite Timoshenko beam resting on identical supports at periodical intervals as shown in Fig. 1. The beam is subjected to the moving forces Q_j characterized by the distance to the first force D_j. Let $R_n(t)$ be the reaction force of a support at the coordinate $x = nl$ (with $n \in \mathbb{Z}$).

In steady-state, we suppose that all supports are equivalent and their responses are described by the same function, but with a delay which equals to the time for the forces to cover the distance between them. In other words, the reaction force repeats when the moving forces pass from one support to another

$$R_n(t) = R\left(t - \frac{nl}{v}\right) \tag{1}$$

where $R(t)$ is the reaction force of the support at the origin of the reference system $x = 0$. The total force applied on the beam can be represented with the help of the Dirac functions

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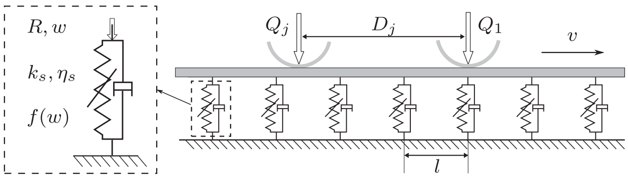

Fig. 1. Forces applied on a periodically supported beam.

$$F(x,t)=\sum_{n=-\infty}^{\infty}R\left(t-\frac{x}{v}\right)\delta(x-nl)-\sum_{j=1}^{K}Q_j\delta(x+D_j-vt) \tag{2}$$

In addition, we have the dynamic equations of the Timoshenko beam

$$\begin{cases}\rho S\frac{\partial^2 w_r}{\partial t^2}=\kappa SG\left(\frac{\partial^2 w_r}{\partial x^2}-\frac{\partial\phi_r}{\partial x}\right)+F(x,t)\\ \rho I\frac{\partial^2\phi_r}{\partial t^2}=EI\frac{\partial^2\phi_r}{\partial x^2}+\kappa SG\left(\frac{\partial w_r}{\partial x}-\phi_r\right)\end{cases} \tag{3}$$

where w_r, ϕ_r are the displacement and the rotation of the section of the beam; ρ, E are the density, the Young's modulus and S, I, κ, G are the section, the inertia, the shear coefficient and the shear modulus of the beam.

Eqs. (2) and (3) define the dynamics of the beam and its supports in the steady-state. Thereafter, we will resolve these equations by performing a Fourier transform with regard to the time t, and then using the Fourier series development with regard to x. Let's denote ∂_t, ∂_x the derivations with regard to t and x. By performing the Fourier transform of Eq. (3) with regard to t, we obtain

$$\begin{cases}\kappa SG\ \partial_x\hat{\phi}_r=\kappa SG\ \partial_x^2\hat{w}_r+\rho S\omega^2\hat{w}_r+\hat{F}\\ -\kappa SG\ \partial_x\hat{w}_r=EI\ \partial_x^2\hat{\phi}_r-(\kappa SG-\rho I\omega^2)\hat{\phi}_r\end{cases} \tag{4}$$

where the hat stands for the Fourier transform with regard to t. Particularly, we obtain the following result from Eq. (2)

$$\hat{F}=\mathrm{e}^{-\frac{\mathrm{i}\omega}{v}x}\left(\hat{R}(\omega)\sum_{n=-\infty}^{\infty}\delta(x-nl)-\sum_{j=1}^{K}\frac{Q_j}{v}\mathrm{e}^{-\frac{\mathrm{i}\omega}{v}D_j}\right) \tag{5}$$

Thus, $\mathrm{e}^{\frac{\mathrm{i}\omega}{v}x}\hat{F}$ is periodic with regard to x. Therefore, if we put

$$\hat{w}_r=\Psi(x,\omega)\mathrm{e}^{-\frac{\mathrm{i}\omega}{v}x}\quad\text{and}\quad\hat{\phi}_r=\Phi(x,\omega)\mathrm{e}^{-\frac{\mathrm{i}\omega}{v}x} \tag{6}$$

Eq. (4) becomes

$$\begin{cases}\kappa SG\left(\partial_x\Phi-\frac{\mathrm{i}\omega}{v}\Phi\right)=\kappa SG\left(\partial_x^2\Psi-2\frac{\mathrm{i}\omega}{v}\partial_x\Psi-\frac{\omega^2}{v^2}\Psi\right)+\rho S\omega^2\Psi+\mathrm{e}^{\frac{\mathrm{i}\omega}{v}x}\hat{F}\\ \kappa SG\left(\frac{\mathrm{i}\omega}{v}\Psi-\partial_x\Psi\right)=EI\left(\partial_x^2\Phi-2\frac{\mathrm{i}\omega}{v}\partial_x\Phi-\frac{\omega^2}{v^2}\Phi\right)-(\kappa SG-\rho I\omega^2)\Phi\end{cases} \tag{7}$$

By the Floquet's theorem [18], Eq. (7) has a periodic solution. We can find this solution by using the Fourier series developments of Φ and Ψ (see Appendix A). Thereafter, by combining the results of Ψ and Φ with Eq. (6) we obtain

$$\begin{cases}\hat{w}_r(x,\omega)=\hat{R}(\omega)\sum_{n=-\infty}^{\infty}\tilde{p}_n\mathrm{e}^{\mathrm{i}x\left(\frac{2\pi n}{l}-\frac{\omega}{v}\right)}-\tilde{p}_0\frac{l}{v}\sum_{j=1}^{K}Q_j\mathrm{e}^{-\frac{\mathrm{i}\omega}{v}(x+D_j)}\\ \hat{\phi}_r(x,\omega)=\hat{R}(\omega)\sum_{n=-\infty}^{\infty}\tilde{q}_n\mathrm{e}^{\mathrm{i}x\left(\frac{2\pi n}{l}-\frac{\omega}{v}\right)}-\tilde{q}_0\frac{l}{v}\sum_{j=1}^{K}Q_j\mathrm{e}^{-\frac{\mathrm{i}\omega}{v}(x+D_j)}\end{cases} \tag{8}$$

where $\tilde{p}_n$, $\tilde{q}_n$ ($n\in\mathbb{Z}$) are the Fourier coefficients of Ψ, Φ calculated by Eq. (A7) in Appendix A. We can reduce Eq. (8) by defining $\eta(x,\omega)$, $\gamma(x,\omega)$ as follows

$$\begin{cases}\eta(x,\omega)=\sum_{n=-\infty}^{\infty}\tilde{p}_n\mathrm{e}^{\mathrm{i}x\left(\frac{2\pi n}{l}-\frac{\omega}{v}\right)}\\ \gamma(x,\omega)=\sum_{n=-\infty}^{\infty}\tilde{q}_n\mathrm{e}^{\mathrm{i}x\left(\frac{2\pi n}{l}-\frac{\omega}{v}\right)}\end{cases} \tag{9}$$

Indeed, η, γ in the last equation can be reduced to simple analytical functions as shown in Eqs. (B11) and (B14) of Appendix B. Then, by substituting Eq. (9) into Eq. (8), we obtain

$$\begin{cases}\hat{w}_r(x,\omega)=\hat{R}(\omega)\eta(x,\omega)-\tilde{p}_0\frac{l}{v}\sum_{j=1}^{K}Q_j\mathrm{e}^{-\frac{\mathrm{i}\omega}{v}(x+D_j)}\\ \hat{\phi}_r(x,\omega)=\hat{R}(\omega)\gamma(x,\omega)-\tilde{q}_0\frac{l}{v}\sum_{j=1}^{K}Q_j\mathrm{e}^{-\frac{\mathrm{i}\omega}{v}(x+D_j)}\end{cases} \tag{10}$$

Eq. (10) is a simple relation between the Fourier transforms of the beam displacement and the reaction force. This is a result of the periodicity condition and the dynamic equation of the Timoshenko beam, which do not depend on the support behaviour. Once the reaction force is calculated, this equation can be used to compute the response of the beam. In the next section, we will introduce a system equivalence based on this relation.

2.2. System equivalence

In order to calculate the reaction force of the support, we need to compute the displacement of the beam at the support position $w(t)=w_r(0,t)$, or its Fourier transform $\hat{w}(\omega)=\hat{w}_r(0,\omega)$. By substituting $x=0$ into Eq. (10), we have

$$\hat{w}_r(0,\omega)=\hat{R}(\omega)\eta(0,\omega)-\tilde{p}_0\frac{l}{v}\sum_j Q_j e^{-i\frac{\omega}{v}D_j} \tag{11}$$

Hence, we can also write

$$\hat{R}(\omega)=\mathscr{K}_T\hat{w}(\omega)+\mathscr{Q}_T \tag{12}$$

where $\mathscr{K}_T$ and $\mathscr{Q}_T$ are defined by

$$\mathscr{K}_T=\eta^{-1}(0,\omega)\quad\text{and}\quad\mathscr{Q}_T=\mathscr{K}_T\tilde{p}_0\frac{l}{v}\sum_{j=1}^{K}Q_j e^{-i\frac{\omega}{v}D_j} \tag{13}$$

with $\tilde{p}_0(\omega)$, $\eta(0,\omega)$ are calculated by Eqs. (A10) and (B15) in the appendices.

Eq. (12) is a linear relation between the force and the displacement applied on the support at $x=0$, and it holds for all supports because of the periodicity condition. This relation is the same as the constitutive law of an equivalent spring with stiffness $\mathscr{K}_T$ and pre-force $\mathscr{Q}_T$. Therefore, we call the system equivalence of a periodically supported Timoshenko beam, which existed also for Euler-Bernoulli beams (see [11]). Eq. (12) explains the distribution mechanism of the moving forces Q_j to the supports via the beam. When a moving force comes toward and leaves away the support along the direction of the beam, the reaction force of the support increases and decreases respectively. This process is the same as a force applied on the support via the equivalent spring.

The comparison of the system equivalences with two parameters stiffness and pre-force for Euler-Bernoulli and Timoshenko beams is shown in Table 1 where the beam parameters correspond to a rail UIC60 [14]. We see that the stiffness $\mathscr{K}$ depends on two parameters $\lambda_{1,2}$ and $C_{1,2}$ which are different between the two beam models. However, if the shear modulus κG and the ratio E/ρ tend to infinity, $\lambda_{1,2}$ and $C_{1,2}$ of the Timoshenko beam tend to the ones of the Euler-Bernoulli beam. Therefore, the stiffness and pre-force for the two beam models are equivalent when the Timoshenko beam does not include the shear modulus κG and the ratio E/ρ. This phenomenon agrees well with the beam theories.

Figs. 2 and 3 show an example of the stiffness and the pre-force for the Timoshenko and the Euler-Bernoulli beams with the parameters presented in Table 2. We see that the two beam models give almost the same equivalent pre-force. Otherwise, the Timoshenko beam gives a smaller equivalent stiffness than the Euler-Bernoulli beam. This difference comes from the fact that the Euler-Bernoulli beam has less degree of freedom. It is remarkable that the difference takes place only at the maximum peaks of the stiffness which correspond to high frequencies. In other words, the influence of the beam models is more important at high frequencies. In the next sections, we will calculate the responses