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On the semi-group of a scaled skew Bessel process

Larbi Alili *, Andrew Aylwin

Department of Statistics, The University of Warwick, CV4 7AL, Coventry, UK

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ABSTRACT

We define scaled skew Bessel processes and determine their Green's functions and semi-group densities. We prove that these processes satisfy the time inversion property although the corresponding densities of the semi-groups are not, in general, twice differentiable in the density and starting point arguments. Some characterizations and semi-martingale properties are given.

1. Introduction

Time inversion was studied in Gallardo and Yor (2005), Lawi (2008), Pitman and Yor (1981), Shiga and Watanabe (1973), Watanabe (1975, 1998a, b); see Alili and Patie (2010) for applications to boundary crossing problems. Skew processes were mostly studied in the settings of Brownian motions and Bessel processes, see Barlow (1988), Blei (2012), Decamps et al. (2006), Harrison and Shepp (1981), Lejay (2006), Ouknine (1990). There is a renewed interest on these processes motivated, for example, by modelling discontinuous media with permeable barriers and related Monte Carlo simulation schemes, see Lejay and Maire (2013). Our main purpose here is to further the skew construction by scaling the underlying Bessel processes in order to obtain a class of continuous self similar Markov processes on $\mathbb{R}$ that satisfy the time inversion property.

We now specify the class of processes we will focus on. Following Harrison and Shepp (1981), and the definition of skew Brownian motions therein, we define a scaled skew Bessel process of dimension $\delta \in (0,2)$, or index $\nu = \frac{\delta}{2} - 1 \in (-1,0)$, with scaling parameters σ_+ and σ_- and skew parameter $0 < p < 1$, or SSBES$(\nu, p, \sigma_-, \sigma_+)$ for short, as a generalized conservative, i.e., with an infinite lifetime, diffusion having scale function

$$S(x) = \begin{cases} \dfrac{1}{p|2\nu|}\left(\dfrac{x}{\sigma_+}\right)^{-2\nu} & \text{if } x > 0; \\ \dfrac{-1}{(1-p)|2\nu|}\left(\dfrac{|x|}{\sigma_-}\right)^{-2\nu} & \text{if } x \leq 0, \end{cases} \quad (1)$$

* Corresponding author.
E-mail address: L.Alili@warwick.ac.uk (L. Alili).

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and speed measure

$$m(\mathrm{d}x)=\begin{cases}2p(\frac{x}{\sigma_+})^{2\nu+1}\frac{dx}{\sigma_+} & \text{if } x>0;\\ 2(1-p)(\frac{|x|}{\sigma_-})^{2\nu+1}\frac{dx}{\sigma_-} & \text{if } x\le 0.\end{cases} \tag{2}$$

A SSBES$(\nu, p, 1, 1)$ is a skew Bessel process or a bilateral Bessel process following the terminology used in Watanabe (1998a). A SSBES process $(R_t, t\geq 0)$ is a continuous strong Markov process that has the scaling (or self-similar or semi-stable) property. That is, for all $c>0$ and $x\in\mathbb{R}$, the process $(c^{-1/2}R_{ct}, t\geq 0)$ with $R_0=\sqrt{c}x$ has the same distribution as $(R_t, t\geq 0)$ with $R_0=x$. We only focus on the cases where $-1<\nu<0$ so that 0 is non-singular for a Bessel process of index ν and for the studied processes; for nice accounts on Bessel processes, we refer to Pitman and Yor (1981) and Revuz and Yor (1999). Of course, SSBES processes do not have the Lamperti–Kiu representation discovered in Chaumont et al. (2013). We shall prove that they satisfy the time inversion property in the sense of Gallardo and Yor, i.e., if R is a SSBES$(\nu, p, \sigma_+, \sigma_-)$ then, for all $x\in\mathbb{R}$, we have

$$(tR_{1/t}, t>0) \text{ is a time-homogeneous Markov process.} \tag{3}$$

Interestingly, their semi-group densities with respect to the Lebesgue measure do not satisfy the twice differentiability condition in the space variables of the settings of Gallardo and Yor (2005) and Lawi (2008).

We shall first establish some characterizations of a SSBES processes in terms of infinitesimal generators and Bessel processes. Next, we explicitly calculate the Green's function and the semi-group densities of a SSBES process. Finally, exploiting results of Gallardo and Yor (2005), we conclude that SSBES semi-groups satisfy the time inversion property. Note that this generalizes the result proved in Graversen and Vuolle-Apiala (2000) that states that symmetric skew Bessel processes satisfy the time inversion property. We also give, when possible, the decompositions of SSBES processes as semi-martingales in their own filtrations. We learnt while preparing this note that the semi-groups of skew Bessel processes, i.e., with $\sigma_\pm=1$, were calculated in the unpublished preprint (Watanabe, 1998a).

2. SSBES processes and their semi-groups

Let $R\sim$ SSBES $(\nu, p, \sigma_+, \sigma_-)$, where $\sigma_+, \sigma_->0$, $0<p<1$ and $-1<\nu<0$. R and other processes considered here are defined on a fixed probability space $(\Omega, \mathcal{F}, \mathbb{P})$. We refer to Theorem VII.3.12 in P. 308 of Revuz and Yor (1999), see also the proof of Lemma 1 of Decamps et al. (2006), for the construction of the infinitesimal generator of a diffusion that is specified by a scale function and a speed measure. Denoting by $\mathcal{A}_R$ the infinitesimal generator of R, writing $\mathcal{A}_R=D_mD_S$, where D_m and D_S are the derivatives with respect to m and S respectively, and using the condition $\mathcal{A}_Rf\in\mathcal{C}_b(\mathbb{R})$, we obtain

$$\mathcal{A}_Rf(x)=\begin{cases}(\sigma_{\mathrm{sgn}(x)})^2\frac{2\nu+1}{2x}f'(x)+\frac{1}{2}(\sigma_{\mathrm{sgn}(x)})^2f''(x) & \text{if } x\neq 0;\\ \lim_{x\uparrow 0}\mathcal{A}_Rf(x)=\lim_{x\downarrow 0}\mathcal{A}_Rf(x) & \text{if } x=0.\end{cases} \tag{4}$$

Since f must be continuously differentiable with respect to S, the domain $\mathcal{D}_R$ of $\mathcal{A}_R$ is

$$\mathcal{D}_R=\left\{f:\mathcal{A}_Rf\in\mathcal{C}_b(\mathbb{R}),\ p\sigma_+^{-2\nu}\lim_{x\downarrow 0}x^{2\nu+1}f'(x)=(1-p)\sigma_-^{-2\nu}\lim_{x\uparrow 0}x^{2\nu+1}f'(x)\right\}. \tag{5}$$

The terminology scaled skew Bessel process comes from the fact that scaled skew Bessel processes can be constructed by scaling in some way a skew Bessel process or the Bessel process involved in the negative and positive parts of the construction of a skew Bessel process, or the Bessel excursions involved in the construction given in Remark 6.

Lemma 1. *The process $X^{(\nu)}:=(R_t/\sigma_{sgn(R_t)}, t\geq 0)$ is a skew Bessel process with skew parameter p and index ν. Moreover, $(|R_t|/\sigma_{sgn(R_t)}, t\geq 0)$ is a Bessel process of index ν.*

Proof. Combining formulae (4) and (5), and the elementary properties of the homeomorphism $\tilde{\sigma}: x\to x/\sigma_{\mathrm{sgn}(x)}$ of $\mathbb{R}$, we get the first assertion. Using Lemma 2.3 of Blei (2012), we obtain the second assertion for the case $\nu\in(-\frac{1}{2}, 0)$. Assume now that $\nu\in(-1, -\frac{1}{2}]$. Set $\phi(x)=2sgn(x)|x|^{1/2}$, $x\in\mathbb{R}$. Define $Y_t:=\phi(X_{\tau_t}^{(\nu/2)})$, where τ_t is the inverse of $\int_0^t|X_s^{(\nu/2)}|^{-1}ds$, $t\geq 0$. Recall that $X^{(\nu/2)}$ is a SSBES$(\nu/2, p, 1, 1)$. Denote by S_1 and m_1 the scale function and the density of its speed measure with respect to the Lebesgue measure, respectively. Since ϕ is a homeomorphism of $\mathbb{R}$, $(Y_t, t\geq 0)$ is a continuous strong Markov process. Its scale function is given by

$$S_1\circ\phi^{-1}(x)=\begin{cases}\frac{1}{2p|\nu|}2^{1+2\nu}(x)^{-2\nu} & \text{if } x>0;\\ \frac{-1}{2(1-p)|\nu|}2^{1+2\nu}(|x|)^{-2\nu} & \text{if } x\le 0,\end{cases}$$