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Lift expectations of random sets

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ABSTRACT

It is known that the distribution of an integrable random vector ξ in $\mathbb{R}^d$ is uniquely determined by a $(d+1)$-dimensional convex body called the lift zonoid of ξ. This concept is generalised to define the lift expectation of random convex bodies. However, the unique identification property of distributions is lost; it is shown that the lift expectation uniquely identifies only one-dimensional distributions of the support function, and so different random convex bodies may share the same lift expectation. The extent of this nonuniqueness is analysed and it is related to the identification of random convex functions using only their one-dimensional marginals. Applications to construction of depth-trimmed regions and partial ordering of random convex bodies are also mentioned.

1. Introduction

Probability theory provides numerous ways of identifying distributions of random variables. Mentioning two (rather nontraditional) examples, the distribution of an integrable random variable ξ is uniquely determined by its stop-loss transform $\mathbf{E}(t+\xi)_+$, $t \in \mathbb{R}$, where x_+ denotes the positive part of $x \in \mathbb{R}$. Furthermore, Hoeffding (1953) showed that the sequence $\mathbf{E}\max(\xi_1,\dots,\xi_n)$, $n \geq 1$, built from i.i.d. copies of ξ uniquely determines the distribution of ξ.

Extensions of these identification results to random vectors are of a geometric nature and rely on the concept of zonoids and zonotopes. Zonotopes form an important family of polytopes, which are defined as Minkowski (elementwise) sums of a finite number of segments. *Zonoids* are convex sets that appear as limits in the Hausdorff metric of a sequence of zonotopes, see Schneider (2014, Sec. 3.5). In the plane, all (centrally) symmetric convex sets are zonoids, while the symmetry is only a strictly necessary condition in dimensions three and more.

Zonoids can also be described as expectations of random segments. For this purpose, recall that a *random convex closed set* X in $\mathbb{R}^d$ is a map from a probability space $(\Omega, \mathfrak{F}, \mathbf{P})$ to the family of convex closed sets in $\mathbb{R}^d$, which is measurable in the sense that $\{\omega : X(\omega)\cap K \neq \emptyset\} \in \mathfrak{F}$ for all compact sets K in $\mathbb{R}^d$, see Molchanov (2017, Def. 1.1.1). If X is almost surely compact and non-empty, it is called a *random convex body*. The measurability condition is then equivalent to the fact that the *support function* of X

$$h_X(u) = \sup\{\langle x, u\rangle : x \in X\}, \qquad u \in \mathbb{R}^d,$$

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is a random function of u, where $\langle x, u\rangle$ is the scalar product. The distribution of a random convex body is uniquely identified by the finite-dimensional distributions of its support function.

A random convex closed set X is said to be *integrable*, if there exists an integrable random vector ξ such that $\xi \in X$ a.s. This vector ξ is called an integrable *selection* of X. The *selection expectation* $\mathbf{E}X$ is the closure of the set of expectations of all its integrable selections, see Molchanov (2017, Sec. 2.1). The closure is not needed if

$$\|X\| = \sup\{\|x\| : x \in X\}$$

is an integrable random variable. Then X is said to be *integrably bounded*.

If X is integrably bounded, then $h_X(u)$ is integrable for all u, and

$$\mathbf{E}h_X(u) = h_{\mathbf{E}X}(u), \qquad u \in \mathbb{R}^d.$$

If X is integrable and ξ is its integrable selection, then $h_X(u) = \langle \xi, u\rangle + h_{X-\xi}(u)$, whence $h_X(u)$ is either integrable or has the well-defined expectation $+\infty$.

If ξ is a random vector in $\mathbb{R}^d$, then the *segment* $[\mathbf{0}, \xi]$ with end-points being the origin $\mathbf{0}$ and ξ is a random convex body. This random convex body is integrable even for a nonintegrable ξ, since it contains the origin. Its expectation $Z_\xi = \mathbf{E}[\mathbf{0}, \xi]$ is called the *zonoid* of ξ, see Mosler (2002) and Molchanov (2017). Often, symmetrised versions of zonoids are defined as expectation of the segment $[-\xi, \xi]$ and assuming the integrability of ξ, see Schneider (2014, Sec. 3.5).

The zonoid does not uniquely determine the distribution of ξ, for instance, it does not change if ξ is multiplied by an independent nonnegative random variable with expectation one. The extent of such nonuniqueness is explored by Molchanov et al. (2014). Despite the nonuniqueness, the zonoid delivers some information about the linear dependence between ξ and η, see Dall'aglio and Scarsini (2003).

It is possible to achieve the uniqueness by uplifting ξ into $\mathbb{R}^{d+1}$. For this, consider the segment $[(0, \mathbf{0}), (1, \xi)]$ in $\mathbb{R}^{d+1}$, and call $\mathbf{E}[(0, \mathbf{0}), (1, \xi)] = \hat{Z}_\xi$ the *lift zonoid* of ξ, see Koshevoy and Mosler (1998) and Mosler (2002). Since

$$h_{\hat{Z}_\xi}(u_0, u) = \mathbf{E}(u_0 + \langle \xi, u\rangle)_+, \quad (u_0, u) \in \mathbb{R}^{d+1},$$

the support function of the lift zonoid is the stop-loss transform of $\langle \xi, u\rangle$. Thus, the lift zonoid of ξ determines uniquely the distribution of the scalar products $\langle \xi, u\rangle$ for all $u \in \mathbb{R}^d$ and so the distribution of ξ. This fact goes back to Hardin (1981), was independently proved by Koshevoy and Mosler (1998), and further gave rise to numerous applications in multivariate analysis, see Mosler (2002).

This paper presents an extension of the lift zonoid concept for random convex bodies. Section 2 defines the required lifting that gives rise to the corresponding expected sets. In general, such a set is no longer a zonoid in the geometric sense of Schneider (2014, Sec. 3.5), and so we call it *lift expectation*. It is shown that the lift expectation of an integrable random convex body X characterises the distributions of the support function of X in any single direction u such that $h_X(u)$ is integrable. Equivalently, the lift expectation embodies the information contained in the one-dimensional marginals of a random convex function.

Examples of lift expectations are provided in Section 3. Section 4 relates the uniqueness issue to the multivariate comonotonicity of the support function. Section 5 deals with random sets having at most a finite number of realisations. Section 6 discusses an extension of this concept for n-tuples of random sets. A numerical example is given in Section 7.

The range of possible applications of the lift expectation is similar to those well-established for lift zonoids, see Mosler (2002), e.g., to assessing the depth of set-valued observations and stochastic ordering of random convex bodies. In particular, the lift expectation can be used to identify outliers in samples of random convex bodies — such samples arise in applications to partially identified problems in econometrics, see Molchanov and Molinari (2018). A relation to risk measures is mentioned in Example 4.1.

2. Lift expectation of a random set

2.1. Univariate distributions of the support function

Let X be a random convex body in $\mathbb{R}^d$. Uplift it to $\mathbb{R}^{d+1}$ by letting

$$Y = \operatorname{conv}(\{0, \mathbf{0}\}, \{1\} \times X) \tag{1}$$

be the convex hull of the origin $(0, \mathbf{0})$ in $\mathbb{R}^{d+1}$ and the set $\{1\} \times X$. The random convex body Y is always integrable; it is integrably bounded if and only if X is integrably bounded.

Definition 2.1. The set $\mathbf{E}Y$ (that is, the selection expectation of Y) is called the *lift expectation* of X and denoted by $\hat{Z}_X$.

The following result establishes that the lift expectation provides exactly the same information as the distributions of the support function $h_X(u)$ for any given u such that $h_X(u)$ is integrable.

Theorem 2.1. *Assume that X and X' are integrable random convex bodies. Then $\hat{Z}_X = \hat{Z}_{X'}$ if and only if the distributions of $h_X(u)$ and $h_{X'}(u)$ coincide for all u such that at least one of them is integrable.*