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Stein operators for variables form the third and fourth Wiener chaoses

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ABSTRACT

Let Z be a standard normal random variable and let H_n denote the nth Hermite polynomial. In this note, we obtain Stein equations for the random variables $H_3(Z)$ and $H_4(Z)$, which represent a first step towards developing Stein's method for distributional limits from the third and fourth Wiener chaoses. Perhaps surprisingly, these Stein equations are fifth and third order linear ordinary differential equations, respectively. As a warm up, we obtain a Stein equation for the random variable $aZ^2 + bZ + c$, $a, b, c \in \mathbb{R}$, which leads us to a Stein equation for the non-central chi-square distribution. We also provide a discussion as to why obtaining Stein equations for $H_n(Z)$, $n \geq 5$, is more challenging.

1. Introduction

1.1. Background

In 1972, Stein (1972) introduced a powerful technique for deriving distributional bounds for normal approximation. Stein's method for normal approximation rests on the following characterisation of the normal distribution: $W \sim N(0,1)$ if and only if

$$\mathbb{E}[f'(W) - Wf(W)] = 0 \tag{1.1}$$

for all real-valued absolutely continuous functions f such that $\mathbb{E}|f'(Z)| < \infty$ for $Z \sim N(0,1)$. This characterisation leads to the so-called Stein equation:

$$f'(x) - xf(x) = h(x) - Nh, \tag{1.2}$$

where Nh denotes $\mathbb{E}h(Z)$ for $Z \sim N(0,1)$, and the test function h is real-valued. It is straightforward to verify that $f(x) = \mathrm{e}^{x^2/2}\int_{-\infty}^{x}[h(t) - Nh]\mathrm{e}^{-t^2/2}\,\mathrm{d}t$ solves (1.2), and bounds on the solution and its derivatives in terms of the test function h and its derivatives are given in Chen et al. (2011) and Döbler et al. (2017). Evaluating both sides of (1.2) at a random variable W and taking expectations gives

$$\mathbb{E}[f'(W) - Wf(W)] = \mathbb{E}h(W) - Nh. \tag{1.3}$$

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Thus, the problem of bounding the quantity $|\mathbb{E}h(W) - Nh|$ has been reduced to bounding the left-hand side of (1.3). For a detailed account of the method see the book (Chen et al., 2011).

In recent years, one of the most significant applications of Stein's method for normal approximation has been to Gaussian analysis on Wiener space. This body of research was initiated by Nourdin and Peccati (2009), in which Stein's method and Malliavin calculus are combined to derive a quantitative fourth moment theorem for the normal approximation of a sequence of random variables living in a fixed Wiener chaos. A detailed account of normal approximation by the Malliavin–Stein method can be found in the book (Nourdin and Peccati, 2012).

One of the advantages of Stein's method is that the above procedure can be extended to treat many other distributional approximations; examples include the Poisson (Chen, 1975), gamma (Gaunt et al., 2017; Luk, 1994), exponential (Chatterjee et al., 2011; Peköz and Röllin, 2011) and variance-gamma distributions (Gaunt, 2014). The Malliavin–Stein method is also applicable to other limits, such as the multivariate normal (Nourdin et al., 2010), exponential and Pearson families (Eden and Viens, 2013; Eden and Viquez, 2015), centered gamma (Döbler and Peccati, 2018; Nourdin and Peccati, 2009), variance-gamma (Eichelsbacher and Thäle, 2015), linear combinations of centered chi-square random variables (Arras et al., 2018, 2017; Azmooden et al., 2015; Nourdin and Poly, 2012), as well as a large family of distributions, such as the uniform, log-normal and Pareto distributions, that satisfy a diffusive assumption (Kusuoka and Tudor, 2012). These works have resulted in analogues of the celebrated fourth moment theorem for non-normal limits.

However, despite these advances, there are many important distributional limits that fall outside the current state of the art of the Malliavin–Stein method. As noted by Peccati (2014), an important class of limits for which little is understood are those of the type $P(Z)$, where $Z \sim N(0, 1)$ and P is polynomial of degree strictly greater than 2. In particular, the case that P is a Hermite polynomial is of particular interest, due to their fundamental role in Gaussian analysis and Malliavin calculus.

Let $H_n(x)$ be the nth Hermite polynomial, defined by $H_n(x) = (-1)^n e^{x^2/2} \frac{d^n}{dx^n}(e^{-x^2/2})$, $n \geq 1$, and $H_0(x) = 1$. The first six Hermite polynomials are then

$$H_1(x) = x, \quad H_2(x) = x^2 - 1, \quad H_3(x) = x^3 - 3x, \quad H_4(x) = x^4 - 6x^2 + 3,$$
$$H_5(x) = x^5 - 10x^3 + 15x, \quad H_6(x) = x^6 - 15x^4 + 45x^2 - 15.$$

In this note, we consider the problem of extending Stein's method to the random variables $H_n(Z)$. Of course, the case $n = 1$ is very well understood. The case $n = 2$ corresponds to the centered chi-square random variable $Z^2 - 1$. This is a special case of the centered gamma distribution, for which the Malliavin–Stein method is also highly tractable; see Azmoodeh et al. (2014, 2016), Döbler and Peccati (2018) and Nourdin and Peccati (2009). As with Stein's method for normal approximation, at the heart of Stein's method for $H_2(Z) = Z^2 - 1$ is the Stein equation

$$2(1 + x)f'(x) - xf(x) = h(x) - N_2h, \tag{1.4}$$

where N_ih denotes the quantity $\mathbb{E}[h(H_i(Z))]$, $i \geq 1$. The Stein equation (1.4) follows by applying a simple translation to the classic gamma Stein equation of Diaconis and Zabell (1991) and Luk (1994). Estimates for the solution of the gamma Stein equation (Döbler et al., 2017; Döbler and Peccati, 2018; Gaunt et al., 2017; Luk, 1994) can then be used to bound the solution of (1.4) and its derivatives. The problem of approximating a random variable W by $H_2(Z)$ is thus reduced to the tractable one of bounding the quantity $\mathbb{E}[2(1 + W)f'(W) - Wf(W)]$.

Recently, Arras et al. (2017) obtained a Stein equation for random variables of the form $F_\infty = \sum_{i=1}^{q}\alpha_i(Z_i^2 - 1)$, where the α_i are real-valued constants and $Z_1, \ldots, Z_q$ are independent $N(0, 1)$ random variables. Their Stein equation for F_∞ was qth order if the α_i are all distinct, and to date no bound exists for the solution of the Stein equation when $q \geq 3$; the case $q = 2$ corresponds to the variance-gamma Stein equation, and bounds from Döbler et al., (2017) and Gaunt, (2014) can be used. Despite not being able to bound the solution of the Stein equation in general, their Stein equation motivated a Stein kernel for F_∞ that was used in Arras et al. (2018) to bound the 2-Wasserstein distance between a random variable F belonging to the second Wiener chaos and the limit F_∞.

1.2. Summary of results

In this note, we consider the problem of extending Stein's method to $H_n(Z)$, $n \geq 3$. In particular, we study the problem of finding Stein equations for $H_n(Z)$, $n \geq 3$. Indeed, our main results are the following Stein equations (see Propositions 2.4 and 2.5) for $H_3(Z)$ and $H_4(Z)$:

$$\begin{aligned}&486(4 - x^2)f^{(5)}(x) - 486xf^{(4)}(x) - 27(8 - x^2)f^{(3)}(x)\\&\quad + 99xf''(x) + 6f'(x) - xf(x) = h(x) - N_3h,\end{aligned} \tag{1.5}$$

and

$$\begin{aligned}&192(x + 6)(3 - x)f^{(3)}(x) + 16(x + 3)(x - 12)f''(x)\\&\quad + 4(11x + 6)f'(x) - xf(x) = h(x) - N_4h.\end{aligned} \tag{1.6}$$

The first striking feature of the Stein equations (1.5) and (1.6) is that they are fifth and third order linear differential equations, respectively. There is a dramatic increase in complexity from the Stein equations (1.2) and (1.4) for $H_1(Z)$ and $H_2(Z)$ to the