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Optimal rate for covariance operator estimators of functional autoregressive processes with random coefficients

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ABSTRACT

We improve a result of Allam and Mourid (2014) by deriving the optimal $\sqrt{n}$ rate for the empirical covariance operators of a Hilbert-valued autoregressive process with random coefficients. Our approach is based on a suitable autoregressive representation of a sequence of covariance operators related to the model, which leads to a decomposition with Hilbert-valued martingale differences. Using large deviation inequalities for Hilbert-valued martingale differences, we then establish exponential bounds and derive the almost sure convergence of the empirical covariance operators in the Hilbert–Schmidt norm, achieving the parametric rate $\sqrt{n}$ up to a $\ln(n)$ factor in the bounded process case.

1. Introduction

In many areas of science such as economics, finance, and medicine, measurements made at several consecutive time points or during a continuous time interval provide a stream of data which can be conveniently described as realizations of random curves or of a functional random variable. These data can then be studied through Functional Data Analysis (FDA), which was gradually developed and has now become a major topic in modern statistics.

Landmark publications in FDA include the pioneering work of Grenander [17], who described insightful rules to approach statistical inference with functional data, and the book by Ramsay and Silverman [26], which covers many practical aspects of FDA. Other key references include [3,13,20]; see also [9,14,16,23,27] for recent overviews and surveys. Previous Special Issues on FDA have appeared in *Statistica Sinica* (vol. 14, no. 3, 2004), *Computational Statistics & Data Analysis* (vol. 51, no. 10, 2007), and the *Journal of Multivariate Analysis* (vol. 101, no. 2, 2010).

The class of functional autoregressive processes has been used to model and predict continuous time random processes. The general theory of functional autoregressive processes was described by Bosq in [3], where various applications are also presented. Examples include the prediction of electricity consumption, road traffic, and El Niño temperatures. Additional applications involving environmental data can be found in [24].

This paper is concerned with the class of functional autoregressive processes with random coefficients, where the autoregressive equation is ruled by random operators. Multivariate time series with random coefficients are used to handle possible nonlinear features of data. This class includes nonlinear models, threshold models, switching models, and some doubly stochastic time series. The corresponding estimation problems were developed, e.g., in [19,25,28].

It is interesting to consider functional autoregressive processes with random coefficients in order to deal with possible nonlinear features in functional data. Guillas [18] proposed a model with exogenous information through a linear operator where the random coefficients take two operator values as per a Bernoulli law. Cugliari [10] generalized such models,

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particularly for cases involving several regimes. He described an application using the national demand for electricity in France (where the temperature information is an exogenous covariate) to express the sensitivity of the demand to meteorological conditions. This relationship is not linear, and it depends on the hour, day, and month of the cold season. In a working paper, Boukhiar and Mourid [5] also applied this model to data, offering competitive results compared with others functional predictors.

Here we consider the estimation of covariance operators and their eigen-elements in the framework of functional autoregressive processes with random coefficients, where the estimation methods involve ill-posed linear inverse problems and perturbation theory; see [3,12,21]. In a previous paper [1], we provided an estimation of covariance operators with convergence rate $n^{1/4}$ (up to a $\ln(n)$ factor) in the Hilbert–Schmidt norm of operators. Our aim here is to improve this result to achieve the parametric rate $\sqrt{n}$ up a $\ln(n)$ factor and in the bounded process case.

Our approach is based on a suitable autoregressive representation of a sequence of covariance operators related to the model, leading to a decomposition with Hilbert-valued martingale differences. We then apply the large deviation inequalities for Hilbert-valued martingale differences to establish exponential inequalities; see, e.g., Theorem 2.14 in [3]. As a consequence, we derive the almost sure convergence of the empirical covariance operators in the Hilbert–Schmidt norm at the parametric rate $\sqrt{n}$ (up a $\ln(n)$ factor), and we then give the optimal rate. The linear structure of model (1) allows us to derive this optimal rate even though a model with a sequence of random coefficients requires more development than a model with only one deterministic coefficient. Clearly, this method cannot be used directly to obtain similar rates in nonparametric or semiparametric functional time series analysis, such as partial linear functional time series [2], single functional index modeling [15], and projection pursuit functional regression [8].

It is worth noting that in statistical inference for functional data (principal components analysis, regression, etc.), polynomial rates n^{γ} with $\gamma \in (0, 1/2]$ are typically obtained; see, e.g., [22] or Corollary 4.1 in [3]. However, it is well known that to achieve the parametric rate $\sqrt{n}$ (without the $\ln(n)$ factor and for a general process), more regularity conditions are required with respect to the smoothness of the paths of the process (paths in Sobolev spaces, etc.), and a deeper analysis is required on the eigenvalue decay rate of the covariance operators, with likely new arguments to support such results; see [7] and references therein.

The paper is organized as follows. The notations and definitions are introduced in Section 2, and the results are presented in Section 3. All proofs are relegated to Section 4.

2. Notations and definitions

We begin by introducing basic facts. Let $(\Omega, \mathcal{A}, P)$ be a complete probability space and $(H, \|\cdot\|)$ be a real separable Hilbert space with inner product $\langle\cdot,\cdot\rangle$ and norm $\|\cdot\|$. We denote by $\mathcal{L}(H)$ the Banach space of continuous linear operators from H to H equipped with the usual operator norm defined, for all $A \in \mathcal{L}(H)$, by

$$\|A\|_{\mathcal{L}} = \sup_{\|x\|=1} \|Ax\|.$$

Let $(\rho_n, n \in \mathbb{Z})$ be a sequence of measurable random operators defined on $(\Omega, \mathcal{A}, P)$ with values in $\mathcal{L}(H)$ endowed with its Borel σ-field. We consider the H-valued sequence $(X_n, n \in \mathbb{Z})$ of random variables defined on $(\Omega, \mathcal{A}, P)$ which verifies, for all $n \in \mathbb{Z}$, the equation

$$X_n = \rho_n X_{n-1} + \varepsilon_n, \tag{1}$$

where $(\varepsilon_n, n \in \mathbb{Z})$ is H-valued white noise, i.e., a sequence of independent identically distributed H-valued random variables with zero mean and $\mathrm{E}(\|\varepsilon_n\|^2) = \sigma_\varepsilon^2 > 0$, where E denotes expectation with respect to the probability measure P. The sequence $(X_n, n \in \mathbb{Z})$ is then called a Hilbert-valued random coefficients autoregressive process (HRCA).

In the case of $\mathbb{R}^d$-valued processes, the class of autoregressive processes with random coefficients has been studied by many authors, and general sufficient conditions ensuring the existence of such processes are well known. See Brandt [6] for the one-dimensional case and Bougerol and Picard [4] for the multivariate case, where necessary and sufficient conditions are given. In the Hilbert space case, Guillas [18] treated a special class where the random coefficients take two operator values with Bernoulli law under a power condition on the norm of the operator values.

In this paper, we study a general model defined by (1), and we slightly improve the conditions under which the model is well defined. For this purpose, we assume the following conditions on (1):

(A1): The random variables $(\rho_n, n \in \mathbb{Z})$ are independent and identically distributed (iid).
(A2): The two sequences $(\rho_n, n \in \mathbb{Z})$ and $(\varepsilon_n, n \in \mathbb{Z})$ are independent.
(A3): $\mathrm{E}\{\ln(\|\rho_n\|)\} < 0$ for all $n \in \mathbb{Z}$.
(A4): (i) $\Delta = \sup\{\|\rho_n\|_{\mathcal{L}}, n \in \mathbb{Z}\} < 1$ a.s. and (ii) $\mathrm{E}(\|X_0\|^4) < \infty$.

In the case of $\mathbb{R}^d$-valued autoregressive processes with random coefficients, conditions (A1)–(A3) are the best assumptions currently available in the literature to ensure a unique strictly stationary solution of (1). Condition (A4) is only imposed to make the proofs more tractable; note that (A4) (i) implies (A3).

We recall that for an H-valued random variable X with zero mean and $\mathrm{E}(\|X\|^2) < \infty$, the covariance operator C_X of X is defined, for all $x \in H$, by $C_X(x) = \mathrm{E}(\langle X, x\rangle X)$. The operator C_X is symmetric, compact, positive, and nuclear. If we denote by