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Asymmetric equilibria of two nested elastic rings

F. Lombardo[a], A. Goriely[b], G. Napoli[a,*]

[a] Dipartimento di Matematica e Fisica "E. De Giorgi", Università del Salento, Lecce, Italy
[b] Mathematical Institute, University of Oxford, Oxford, UK

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ABSTRACT

The packing of soft elastic structures is an important and challenging problem due to the possibility of multiple discrete and continuous zones of contact between different parts of the material. To address this problem, we consider the simplest possible packing problem of a thin elastic ring confined within another shorter flexible ring. The elastic properties as well as the dimensionality of both structures, combined with the contact condition yield a wide a variety of possible equilibrium shapes. When the rings are assumed to be inextensible and unshearable, the equilibrium shapes depend only on their relative bending stiffness κ, and on their relative length μ. Whereas the symmetric equilibria for such a problem have been completely determined, the possibility of asymmetric equilibria with lower energy has not yet been considered. For a fixed value of the relative bending stiffness, we explore these symmetry-breaking equilibria as the length of the inner ring increases. We show that, for $\mu \simeq 1.9$ there is a symmetry-breaking bifurcation and asymmetric equilibria are preferred in order to relax the elastic energy.

1. Introduction

Many natural and man-made structures are obtained by the constrained packing of materials within a given volume. Therefore, a natural problem is to determine the morphology of confined flexible materials. This problem has many applications in nature such as DNA packaging [23], mitochondria organization [3], the morphology of plant leaves in buds [17] and even in arterial diseases [6,26]. Packing and folding problems play also a key role in optimization and design of flexible devices [16,27]. Further, beyond the elastic regime, thin sheets can develop intricate ridge networks when folded and crumpled [1]. More recently, the problem of confined elastic curves has gained interest amongst mathematicians [7,9,15,18] and control engineers [2].

The confinement of a single flexible membrane or elastic sheet inside a rigid container has been studied extensively, both in two dimensions [4,8,13,21,24] and in three dimensions [5,14]. Whereas most of these studies assume that the container is fixed, the pressure created by the confined material can also deform the container [25,28]. For instance microtubules surrounded by lipid membranes, can deform the membrane significantly [10]. The possibility of deforming the restraining structure creates very rich systems that can exhibit a wide variety of shapes depending on the geometric and material properties. A paradigm for such problem is the case of two nested elastic rings. In [20] we studied the existence of symmetric solutions (with respect to an axis) by increasing the length of the inner ring up to the first point of contact. The energy minimizer within the set of symmetric solutions was identified and studied as a function of the relative stiffness and length. The effect of adhesion and external pressure was also studied. However, this study did not consider the solution after the first self contact and was restricted to symmetric solutions. The present work addresses both issues and it generalizes the analysis of Boue and co-workers [4,24] by taking into account the flexibility of the container. The main result is the existence of a generic symmetry-breaking bifurcation where asymmetric shape become global minimizers of the problem. Such non-symmetric minimizers in self-contact problems are also known in other elastic problems [4,11,19,22].

2. The model

We consider the planar equilibria of a one-dimensional elastic ring constrained in another shorter elastic ring. We assume that the rings are inextensible and unshearable and neglect friction. Fixing, without loss of generality, the length of the outer ring to be one, we use the length of the inner ring μ as the main control parameter and think of an increase in length as a growth process. Generically, the geometry of the problem includes portions where the two rings are in contact or the inner ring touches itself. These regions evolve as the relative length changes. Thus, regions of the rings that are initially separated may end up in contact during the

* Corresponding author.
E-mail address: gaetano.napoli@unisalento.it (G. Napoli).

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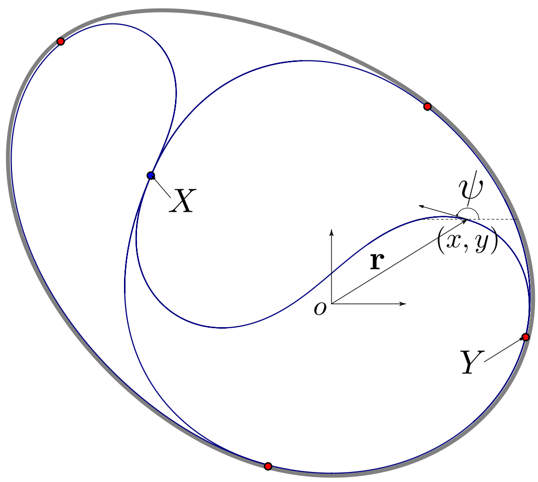

Fig. 1. Schematic representation of an equilibrium shape. The $Y-$nodes are located in correspondence of the red dot, while the $X-$node is represented by the blue dot. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

growth-induced packing process. We suppose that when material points are in contact they have a common position in the plane. Thus, stack of rods that are in contact in a finite region can be described by a planar *elastica*, endowed with a suitable bending stiffness. Consequently, the structure can be understood as a closed graph with inextensible elastic edges of unknown length. There are two kinds of nodes: (i) $Y-$nodes where a stack of layers in contact bifurcates tangentially into two groups, and (ii) $X-$nodes at which two rods meet at a single point (see Fig. 1).

Following the notation of Goriely [12], a point p on the curve can be parameterized by its Cartesian coordinates

$$\mathbf{r}(s) = x(s)\mathbf{e}_x + y(s)\mathbf{e}_y, \tag{1}$$

where s is the arc length. Denoting $\psi(s)$ the angle between the tangent $\boldsymbol{\tau}(s)$ and the horizontal axis $\mathbf{e}_x$, we have

$$\boldsymbol{\tau}(s) \equiv \mathbf{r}'(s) = \cos\psi(s)\mathbf{e}_x + \sin\psi(s)\mathbf{e}_y, \tag{2}$$

where a prime denotes the differentiation with respect to s. By replacing (1) into (2), we obtain:

$$x'(s) = \cos\psi(s), \qquad y'(s) = \sin\psi(s), \tag{3}$$

that should be solved together with the mechanical balance equations given below.

Each rod must obey the balance of angular momentum that, in the absence of external distributed torques, reads

$$\mathbf{m}'(s) + \boldsymbol{\tau}(s) \times \mathbf{n}(s) = \mathbf{0}, \tag{4}$$

where $\mathbf{m}$ and $\mathbf{n}$ are the resultant couple and force, respectively. Moreover, the balance of linear momentum in the absence of external distributed loads assures that $\mathbf{n}$ is constant along each edge. According to the Euler–Bernoulli theory of rods, the internal moment for a planar deformation is proportional to the curvature $c = \mathrm{d}\psi/\mathrm{d}s$, so that:

$$\mathbf{m}(s) = kc(s)\mathbf{e}_z, \tag{5}$$

where k is the bending stiffness and $\mathbf{e}_z = \mathbf{e}_x \times \mathbf{e}_y$. We denote by k_- and k_+ the bending stiffness of the inner and of the outer ring, respectively. If the edge represents a stack of rods then k is the sum of the bending stiffnesses of all rods in the stack. Thus, (4) reduces to the second order ordinary differential equation

$$k\psi''(s) - n_x \sin\psi(s) + n_y \cos\psi(s) = 0, \tag{6}$$

where n_x and n_y, the horizontal and vertical components of $\mathbf{n}$, are among the unknowns of the problem.

Boundary conditions are imposed at the nodes whose locations are *a priori* unknowns. At a Y-node, a mother edge bifurcates into two daughter edges (each of these edges could be a single rod or a stack of rods). These groups must obey the balance [20]

$$\mathbf{n} + \mathbf{n}_1 + \mathbf{n}_2 = \mathbf{0}; \tag{7a}$$

$$\psi' = \psi_1' = \psi_2', \tag{7b}$$

where quantities without subscript refer to the *mother edge* (the adhered region), while quantities with subscripts 1 and 2 refer to the daughter edges. Eq. (7a) represents the balance of the force at the node, while (7b) expresses the continuity of the curvature at the detachment point. Note that (7b) implies that $x(s)$, $y(s)$, and $\psi(s)$ are continuous at the node, for each curve.

At a X-node, two edges meet at an isolated point. Again, we require that each ring has continuous curvature at the node:

$$[\![\psi_1']\!] = [\![\psi_2']\!] = 0, \tag{8}$$

where $[\![\cdot]\!]$ denotes the jump of a quantity at the node relative to the same rod. Further, since the contact is assumed to be frictionless the rods can freely glide on each other. This implies that the tangential component of $\mathbf{n}$ is continuous for each edge. However, the (unilateral) contact constraint induces a jump of the normal component of $\mathbf{n}$ at the node. The force conditions read:

$$[\![\mathbf{n}_1]\!] \cdot \boldsymbol{\tau} = [\![\mathbf{n}_2]\!] \cdot \boldsymbol{\tau} = 0, \tag{9a}$$

$$([\![\mathbf{n}_1]\!] + [\![\mathbf{n}_2]\!]) \cdot \boldsymbol{\nu} = 0, \tag{9b}$$

where $\boldsymbol{\nu} = \mathbf{e}_z \times \boldsymbol{\tau}$ is the normal to the curve.

Finally, we have to satisfy two global constraints: the sums of the lengths of edges belonging to the inner (resp. outer) rings must be L_- (resp. $L_+ = 1$), where L_- and L_+ are the lengths of the inner and outer loop, respectively.

3. Results

We are interested in computing the shape as a function of the two dimensionless parameters

$$\mu := \frac{L_-}{L_+} \qquad \text{and} \qquad \kappa := \frac{k_+}{k_-}.$$

An increase in μ corresponds to a lengthening of the inner loop and in increase in κ represents a (relative) stiffening of the outer loop. We examine the shapes for three different representative values of κ: $\kappa = 10$ (very stiff container), $\kappa = 1$ (same stiffness), and $\kappa = 0.1$ (very flexible container).

We solve the problem numerically by using the Matlab routine `bvp4c`. For each edge of length ℓ, there are *seven unknowns* $\{x(s), y(s), \psi(s), \psi'(s), n_x, n_y, \ell\}$. The unknowns $\{x(s), y(s), \psi(s), \psi'(s)\}$ are related to four first order differential equations: the two kinematic relations (3) and the balance of the angular momentum (6) (notice that (6) can be written as a system of two first order differential equations). The remaining three unknowns $\{n_x, n_y, \ell\}$ are constant parameters that can be determined by the boundary conditions together with the use of the global constraint on the rings length.

A simple count reveals that there are 14 boundary conditions at a X-node and 10 for a Y-node. However, the boundary conditions at the nodes are not sufficient to obtain a unique solution since the equations are invariant under rigid transformations (planar translations and rotations about $\mathbf{e}_z$). From a computational point of view, it is therefore necessary to introduce a fictitious node, that we call the *fixed point*. Thus, the introduction of the fixed point increases the number of edges by one. At the fixed point, we assign its position (two boundary conditions) and the orientation of the curve