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An efficient two-level algorithm for the 2D/3D stationary incompressible magnetohydrodynamics based on the finite element method

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ABSTRACT

In the paper, we develop a new two-level finite element algorithm for solving the 2D/3D stationary incompressible magnetohydrodynamics based on the Newton iterative method. This algorithm is consisting of solving one nonlinear system on a coarse mesh with mesh size H and two linearized problems with different loads on a fine mesh with mesh size h. Compared with existing work on the two-level method for the MHD model, our two-level method allows a much high order scaling between the coarse and fine grid sizes. Furthermore, stability and convergence of this present method are analyzed. Finally, the applicability and effectiveness of the present algorithm are illustrated by several numerical experiments.

1. Introduction

The stationary incompressible magnetohydrodynamics (MHD) system models the motion of electrically conducting, incompressible viscous flows in the presence of an external magnetic field, which is governed by the Navier-Stokes equations of hydrodynamics and Maxwell equations of electromagnetism coupled with the Lorentz's force and Ohm's law. This model has much applications in many fields of industry and life, such as stirring of liquid metal, process metallurgy, electromagnetic pumping, MHD generators and so on.

In this paper, we consider stationary incompressible MHD model as follows ([15,16]):

$$\begin{aligned} -R_e^{-1}\Delta u + u\cdot\nabla u + \nabla p - S\mathrm{curl}B\times B &= f \quad \text{in } \Omega,\\ SR_m^{-1}\,\mathrm{curl}(\mathrm{curl}\,B) - S\,\mathrm{curl}(u\times B) &= g \quad \text{in } \Omega,\\ \nabla\cdot u &= 0 \quad \text{in } \Omega,\\ \nabla\cdot B &= 0 \quad \text{in } \Omega, \end{aligned} \tag{1}$$

with boundary conditions

$$u = 0, \qquad B\cdot n = 0, \qquad n\times \mathrm{curl}\,B = 0, \qquad \text{on } \partial\Omega, \tag{2}$$

where $\Omega \subset R^d$ ($d = 2$ or 3) is a convex polygonal/polyhedral domain, u velocity field, B magnetic field, p pressure, R_e hydrodynamic Reynolds number, R_m magnetic Reynolds number, S coupling number, n outward normal unit vector of $\partial\Omega$ and f and g are external force terms. Here, $u = (u_1, u_2, 0)$, $B = (B_1, B_2, 0)$, $f = (f_1, f_2, 0)$, $g = (g_1, g_2, 0)$ for $d = 2$, and $u = (u_1, u_2, u_3)$, $B = (B_1, B_2, B_3)$, $f = (f_1, f_2, f_3)$, $g = (g_1, g_2, g_3)$ for $d = 3$.

In the last two decades, there has been a rapid development in numerical methods for solving the MHD model [1,4,5,14,19,24,26,28]. Particularly, some stabilized finite element formulations for the MHD equations had been proposed by Badia et al. [3] and Salah et al. [25]. In [17], mixed finite element approximation of incompressible MHD problem in non-convex domains based on weighted regularization was introduced and analyzed. Convergence analysis of three finite element iterative methods for the 2D/3D stationary incompressible MHD problem was studied [11].

As is known, the two-level discretization [29,30] is an important method to reduce computational cost and improve accuracy of finite element solutions. By using this two-level strategy, Layton et al. [23] have described and analyzed a two-level finite element method for discretizing the MHD equations, which involves solving a small nonlinear problem on a coarse mesh and then one large linear problem on a fine mesh. Further, a stabilizing subgrid method combined with the two-level finite element method is described and applied to the MHD Eqs. [2]. Zhang et al. [31] have proposed some two-level coupled correction and decoupled parallel correction finite element methods for solving the considered equations. Recently, Dong and He [9] have presented a two-level Newton iterative method for the 2D/3D stationary incompressible MHD system. This strategy is solving a nonlinear problem on a coarse mesh H, then solving a linear problem on a fine mesh $h = O(H^2)$ based on the MINI-element, and it can save a large

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computation time compared with the one-level method. However, it is known that the two-level method is considered to be more effective for the case $h \ll H$, in particular for 3D problem.

Hence, it is important to find an efficient algorithm to increase the ratio between H and h of the two-level method. Recently, Dai and Cheng [12] have show a two-grid method for solving the Navier-Stokes equations based on Newton iteration. This method involves solving one small nonlinear system on a coarse mesh and two large linear problems on the fine mesh, which allows a much higher order scaling between the coarse grid size and fine grid size. Further, for the stream function formulation of the stationary Navier-Stokes Eqs. [27], the transient Navier-Stokes Eqs. [8], the Kelvin-Voigt model (one more term than the transient Navier-Stokes equations) [6,7], the coupled Navier-Stokes and Darcy system [20] and the natural convection Eqs. [21], the proposed two-level schemes have also dramatically raised the ratio of the mesh size between the coarse and fine grids.

This paper focuses on two-level method for the 2D/3D the stationary incompressible MHD model based on Newton iteration. The main purpose is to improve scaling between the coarse grid size and fine grid size, comparing with the existing work on the two-level method based on Newton iteration [9]. In this article, we will extend the algorithm of Dai and Cheng [12] to the 2D/3D stationary incompressible MHD model. It can be cast in the framework of Dai and Cheng. However, this paper is different from [12] because of the different and more complicated equations. Three fields in the MHD equations which are interactional and interdependent make scaling between the coarse and fine grids harder to establish than that of the Navier-Stokes equations in [12]. The remainder of this article is structured as follows: in Section 2, we introduce some basic notations and results of problem (1)–(2), and recall stability and convergence of standard finite element for the MHD equations. In next section, we present a new two-level method for MHD equations, and give stability and error estimates. In Section 4, some numerical tests confirm the effectiveness of our algorithm.

2. Preliminaries

For a positive integer m, we denote by $H^m(\Omega)$ the Hilbert space of $L^2(\Omega)$ functions whose distributional derivatives up to order m are in $L^2(\Omega)$. The space ${H_0}^m(\Omega)$ consists of $H^m(\Omega)$ functions with vanishing trace up to order $m-1$, and ${L_0}^2(\Omega)$ consists of square integrable functions with vanishing mean. The norm of $H^m(\Omega)$ is denoted by $\|\cdot\|_m$, and the $L^2(\Omega)$ norm and inner product are given by $\|\cdot\|_m$ and $(\cdot,\cdot)$, respectively.

Further, we set the shorthand notation

$$\begin{aligned}
&X = H_0^1(\Omega)^d = \{v \in H^1(\Omega)^d\colon v = 0 \ \text{ on } \partial\Omega\},\\
&W = H_n^1(\Omega)^d = \{\Psi \in H^1(\Omega)^d\colon \Psi\cdot n = 0 \ \text{ on } \partial\Omega\},\\
&V = \{v \in X\colon \operatorname{div} v = 0 \ \text{ in } \Omega\},\\
&V_n = \{\Psi \in W\colon \operatorname{div}\Psi = 0 \ \text{ in } \Omega\},\\
&M = L_0^2(\Omega) = \left\{q \in L^2(\Omega)\colon \int_\Omega q dx = 0\right\}.
\end{aligned}$$

For convenience, we set the product space $W_{0n} = X \times W$ endowed with the usual norm $\|(v,\Psi)\|_1$, where $\|(v,\Psi)\|_i = (\|v\|_i^{\,2} + \|\Psi\|_i^{\,2})^{1/2}$ for all $v \in H^i(\Omega)^d \cap X$, $\Psi \in H^i(\Omega)^d \cap W$, $i = 0,1,2$. And define the dual space $H^{-1}(\Omega)$ of ${H_0}^1(\Omega)$ with the following norm

$$\left\|f\right\|_{-1} = \sup_{v\in X, v\neq 0}\frac{\langle f,v\rangle}{\|v\|_1},$$

where $\langle\cdot,\cdot\rangle$ denotes duality product between the function space X and its dual.

Multiplying (1) by appropriate test functions and integrating (by parts) over the domain Ω in the usual way, we obtain the variational formulation of the MHD system (1), i.e., find $((u,B),p) \in W_{0n} \times M$ such that for all $((v,\Psi),q) \in W_{0n} \times M$,

$$A_0((u,B),(v,\Psi)) + A_1((u,B),(u,B),(v,\Psi)) - d((v,\Psi),p) + d((u,B),q) = \langle F,(v,\Psi)\rangle, \tag{3}$$

where

$$\begin{aligned}
A_0((u,B),(v,\Psi)) &= R_e^{-1}(\nabla u,\nabla v) + SR_m^{-1}(\nabla\times B,\nabla\times\Psi)\\
&\quad + SR_m^{-1}(\nabla\cdot B,\nabla\cdot\Psi),\\
A_1((u,B),(w,\Phi),(v,\Psi)) &= \frac{1}{2}(u\cdot\nabla w,v) - \frac{1}{2}(u\cdot\nabla v,w) - S(\operatorname{curl}\Phi\times B,v)\\
&\quad + S(\operatorname{curl}\Psi\times B,w),\\
d((u,B),q) &= (\nabla\cdot u,q), \qquad \langle F,(v,\Psi)\rangle = \langle f,v\rangle + (g,\Psi).
\end{aligned}$$

It is easy to verify that following properties hold [10,16,22]: for all (u,B), (w,Φ), $(v,\Psi) \in W_{0n}$,

$$A_0((w,\Phi),(w,\Phi)) \geq \nu\|(w,\Phi)\|_1^2, \tag{4}$$

$$A_1((u,B),(w,\Phi),(w,\Phi)) = 0, \tag{5}$$

$$A_1((u,B),(w,\Phi),(v,\Psi)) \leq N\|(u,B)\|_1\|(w,\Phi)\|_1\|(v,\Psi)\|_1, \tag{6}$$

$$A_1((u,B),(w,\Phi),(v,\Psi)) \leq CN\|(u,B)\|_1^{\varepsilon}\|(u,B)\|_0^{1-\varepsilon}\|(w,\Phi)\|_1\|(v,\Psi)\|_1, \tag{7}$$

where $\nu = \min\{{R_e}^{-1}, SC_1{R_m}^{-1}\}$, $N = \sqrt{2}C_0^2\max\{1,\sqrt{2}S\}$, $\varepsilon > 0$ is arbitrarily small for $d = 2$ and $\varepsilon = \frac{1}{2}$ for $d = 3$, and C (with or without a subscript) denotes a generic positive constant, which is independent of the mesh size, but may depend on Ω and other parameters introduced in this paper.

Further, we recall the following existence and uniqueness of solution of (3) in [11,16]:

Theorem 2.1. *The problem* (3) *exists at least a solution pair* $((u,B),p) \in W_{0n} \times M$ *which satisfies*

$$\nu\,\|(u,B)\|_1 \leq \|F\|_{-1}, \tag{8}$$

where $\left\|F\right\|_{-1} = \sup_{(v,\Psi)\in W_{0n}}\frac{\langle F,(v,\Psi)\rangle}{\|(v,\Psi)\|_1}$*.Moreover, if* R_e, R_m*and* S *satisfy the following uniqueness condition*:

$$0 < \sigma = \frac{N\|F\|_{-1}}{\nu^2} < 1, \tag{9}$$

then the solution pair $((u,B),p)$ of problem (3) is unique.

Theorem 2.2. [11] *Suppose that* (8) *and* f, $g \in L^2(\Omega)^d$*are valid, then solution* $((u,B),p)$ *of the problem* (3) *satisfies the following regularity*

$$\nu\,\|(u,B)\|_2 + \|p\|_1 \leq C\|F\|_0. \tag{10}$$

Next, we recall Galerkin finite element method for stationary incompressible MHD problem. Let μ ($\mu = h$ or H with $h \ll H$) be a real positive parameter, and $K_\mu = \{K\colon \cup_{K\subset\Omega}\overline{K} = \overline{\Omega}\}$ be a quasi-uniform partition of Ω into triangles for $d = 2$ or tetrahedra for $d = 3$. Based on the regular partition K_μ, we consider the finite element space pair $(X_\mu, W_\mu, M_\mu) \subset (X, W, M)$ as follow:

$$\begin{aligned}
&X_\mu = (P_{1,\mu}^b)^d \cap X,\\
&W_\mu = (P_{1,\mu}^b)^d \cap W,\\
&M_\mu = \{q_\mu \in C^0(\Omega)\colon q_\mu|_K \in P_1(K), \forall\, K \in K_\mu\},
\end{aligned}$$

where

$$P_{1,\mu}^b = \{v_\mu \in C^0(\Omega)\colon v_\mu|_K \in P_1(K) \oplus \operatorname{span}\{\widehat{b}\}, \forall\, K \in K_\mu\},$$

$\widehat{b}$ is a bubble function, and $P_1(K)$ denotes the space of polynomials on K of degree less than or equal to 1. Let ${W_{0n}}^\mu = X_\mu \times W_\mu$. Then ${W_{0n}}^\mu \times M_\mu$ satisfies the following properties (see [11,18]):

(A1). There exists a constant $\beta > 0$ (depending only on Ω) such that