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Biharmonic functions on spheres and hyperbolic spaces

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ABSTRACT

We construct new explicit proper r-harmonic functions on the standard n-dimensional hyperbolic spaces $\mathbb{H}^n$ and spheres $\mathbb{S}^n$ for any $r \geq 1$ and $n \geq 2$.

1. Introduction

The biharmonic equation is a fourth order partial differential equation which arises in areas of continuum mechanics, including elasticity theory and the solution of Stokes flows. The literature on biharmonic functions is vast, but usually the domains are either surfaces or open subsets of flat Euclidean space $\mathbb{R}^n$.

Recently, new explicit biharmonic functions were constructed on the classical compact simple Lie groups $\mathbf{SU}(n)$, $\mathbf{SO}(n)$ and $\mathbf{Sp}(n)$, see [1]. This gives solutions on the 3-dimensional round sphere $\mathbb{S}^3 \cong \mathbf{SU}(2)$ and the standard hyperbolic space $\mathbb{H}^3$ via a general duality principle. In the literature we have only found explicit proper biharmonic functions from spheres and hyperbolic spaces of dimensions 2 and 3. For this see the papers [1,2] and [3].

The aim of this work is to extend the investigation to higher dimensional spheres $\mathbb{S}^n$ and hyperbolic spaces $\mathbb{H}^n$. We construct a wide collection of new proper biharmonic functions from these spaces of any dimension $n \geq 2$.

The n-dimensional hyperbolic space can be modelled in several different ways. The classical upper-half space model $\mathbb{H}^n$ is the most useful for our purposes. First we construct a wealth of proper r-harmonic functions on the hyperbolic upper-half space $\mathbb{H}^n$, see Theorem 3.3. Then we formulate our solutions in terms of the standard one-sheeted hyperboloid $\mathcal{H}^n$ as a hypersurface of the corresponding Minkowski space M^{n+1}, see Theorem 4.3. Finally we employ a general duality principle, between $\mathcal{H}^n$ and the standard n-dimensional sphere $\mathbb{S}^n$, to construct proper r-harmonic functions on $\mathbb{S}^n$, see Theorem 5.3.

2. Preliminaries

Let (M, g) be a smooth m-dimensional manifold equipped with a Riemannian metric g. We complexify the tangent bundle TM of M to $T^{\mathbb{C}}M$ and extend the metric g to a complex-bilinear form on $T^{\mathbb{C}}M$. Then the gradient ∇f of a complex-valued

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function $f : (M, g) \to \mathbb{C}$ is a section of $T^{\mathbb{C}}M$. In this situation, the well-known *linear* Laplace–Beltrami operator (alt. tension field) τ on (M, g) acts locally on f as follows

$$\tau(f) = \mathrm{div}(\nabla f) = \sum_{i,j=1}^{m} \frac{1}{\sqrt{|g|}} \frac{\partial}{\partial x_j}\left(g^{ij}\sqrt{|g|}\,\frac{\partial f}{\partial x_i}\right). \tag{2.1}$$

Definition 2.1. For an integer $r > 0$ the iterated Laplace–Beltrami operator τ^r is given by

$$\tau^0(f) = f \text{ and } \tau^r(f) = \tau(\tau^{(r-1)}(f)).$$

We say that a complex-valued function $f : (M, g) \to \mathbb{C}$ is

(a) *r-harmonic* if $\tau^r(f) = 0$, and
(b) *proper r-harmonic* if $\tau^r(f) = 0$ and $\tau^{(r-1)}(f)$ does not vanish identically.

It should be noted that the *harmonic* functions are exactly r-harmonic for $r = 1$ and the *biharmonic* functions are the 2-harmonic ones. In some texts, the r-harmonic functions are also called *polyharmonic* of order r.

In the paper [1] the authors develop an interesting connections between the theory of r-harmonic functions and the notion of harmonic morphisms. More specifically, we recall that a map $\pi : (\hat{M}, \hat{g}) \to (M, g)$ between two semi-Riemannian manifolds is a *harmonic morphism* if it pulls back germs of harmonic functions to germs of harmonic functions. The standard reference on this topic is the book [4] of Baird and Wood. We also recommend the updated online bibliography [5]. Later on we will make use of the following result.

Proposition 2.2 (*[1]*). *Let $\pi : (\hat{M}, \hat{g}) \to (M, g)$ be a submersive harmonic morphism from a semi-Riemannian manifold $(\hat{M}, \hat{g})$ to a Riemannian manifold (M, g). Further let $f : (M, g) \to \mathbb{C}$ be a smooth function and $\hat{f} : (\hat{M}, \hat{g}) \to \mathbb{C}$ be the composition $\hat{f} = f \circ \pi$. If $\lambda : \hat{M} \to \mathbb{R}^+$ is the dilation of π then the tension field satisfies*

$$\tau(f) \circ \pi = \lambda^{-2}\tau(\hat{f}) \text{ and } \tau^r(f) \circ \pi = \lambda^{-2}\tau(\lambda^{-2}\tau^{(r-1)}(\hat{f}))$$

for all positive integers $r \geq 2$.

3. The hyperbolic upper-half space $\mathbb{H}^n$

In this section we construct new complex-valued proper r-harmonic functions on the n-dimensional hyperbolic space $\mathbb{H}^n$ for any $r \geq 1$ and $n \geq 2$. We model $\mathbb{H}^n$ as the hyperbolic upper-half space i.e. the differentiable manifold

$$\mathbb{H}^n = \{(t, x) \mid t \in \mathbb{R}^+ \text{ and } x \in \mathbb{R}^{n-1}\}$$

equipped with its standard Riemannian metric ds^2 satisfying

$$ds^2 = \frac{1}{t^2} \cdot (dt^2 + dx_1^2 + \cdots + dx_{n-1}^2).$$

It is then a direct consequence of Eq. (2.1) that the corresponding Laplace–Beltrami operator τ satisfies

$$\tau(f) = t^2 \cdot (\frac{\partial^2 f}{\partial x_1^2} + \cdots + \frac{\partial^2 f}{\partial x_{n-1}^2}) + t^2 \cdot \frac{\partial^2 f}{\partial t^2} - (n-2) \cdot t \cdot \frac{\partial f}{\partial t}.$$

Theorem 3.1. *Let the n-dimensional hyperbolic space $\mathbb{H}^n$ be modelled as the upper-half space i.e. $\mathbb{H}^n = \mathbb{R}^+ \times \mathbb{R}^{n-1}$. Let $h : \mathbb{R}^{n-1} \to \mathbb{C}$ be a non-constant function harmonic with respect to the Euclidean metric on $\mathbb{R}^{n-1}$ and $p_1 : \mathbb{R}^+ \to \mathbb{C}$ be differentiable. Then the function $f_1 : \mathbb{H}^n \to \mathbb{C}$ defined by*

$$f_1(t, x) = p_1(t) \cdot h(x)$$

is harmonic on $\mathbb{H}^n$ if and only if p_1 is of the form $p_1(t) = a_1 + b_1 \cdot t^{n-1}$, for some constants $a_1, b_1 \in \mathbb{C}$.

Proof. We are assuming that $h : \mathbb{R}^{n-1} \to \mathbb{C}$ is a harmonic function with respect to the Euclidean metric on $\mathbb{R}^{n-1}$ i.e.

$$\frac{\partial^2 h}{\partial x_1^2} + \cdots + \frac{\partial^2 h}{\partial x_{n-1}^2} = 0.$$

Then the tension field $\tau(f_1)$ satisfies

$$\tau(f_1) = t^2 \cdot h(x) \cdot \frac{\partial^2 p_1}{\partial t^2} - (n-2) \cdot t \cdot h(x) \cdot \frac{\partial p_1}{\partial t} = h(x) \cdot \tau(p_1).$$

This means that $f_1 : \mathbb{H}^n \to \mathbb{C}$ is harmonic if and only if $\tau(p_1) = 0$, or equivalently,

$$t^{-n} \cdot \tau(p_1) = t^{2-n} \cdot \frac{\partial^2 p_1}{\partial t^2} + (2-n) \cdot t^{1-n} \cdot \frac{\partial p_1}{\partial t} = \frac{\partial}{\partial t}(t^{2-n} \cdot \frac{\partial p_1}{\partial t}) = 0.$$