



The energy equality for weak solutions to the equations of non-Newtonian fluids

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ABSTRACT

In this short paper, we extend the result of Shinbrot (1974) to an incompressible fluid with shear dependent viscosity. It is shown that a weak solution to the equations of non-Newtonian fluids lying in a space $L^q(0, T; L^p)$ satisfies an energy equality, where $\frac{2r}{r-1} \leq p \leq \frac{2r}{r-2}$ and $\frac{1}{p} + \frac{1}{q} \leq \frac{r-1}{r}$, if $r > 2$; $p \geq \frac{2r}{r-1}$ and $\frac{r-1}{p} + \frac{1}{q} = \frac{r-1}{2}$, if $\frac{2(n+1)}{n+2} < r \leq 2$. In particular, our result implies that the weak solution must satisfy the energy equality when $r \geq \frac{3n+2}{n+2}$, which is consistent with the known fact.

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1. Introduction

Let $\Omega \subset \mathbb{R}^n$ ($n \geq 2$) be a domain with a sufficiently smooth boundary. We consider a non-Newtonian incompressible fluid which is governed by the following system

$$\begin{cases} \mathbf{u}_t + \mathbf{u} \cdot \nabla \mathbf{u} - \operatorname{div} (|D(\mathbf{u})|^{r-2} D(\mathbf{u})) + \nabla \pi = 0, & \text{in } \Omega \times (0, T), \\ \operatorname{div} \mathbf{u} = 0, & \text{in } \Omega \times (0, T), \\ \mathbf{u}(0, x) = \mathbf{u}_0(x), & \text{in } \Omega. \end{cases} \quad (1)$$

where $\mathbf{u} = (u_1, u_2, u_3)$ denotes the unknown velocity of the fluid and π the pressure, and

$$D(\mathbf{u}) = \frac{1}{2} (\nabla \mathbf{u} + (\nabla \mathbf{u})^T).$$

We first give the definition of a weak solution to (1). To this end, we denote by $C_0^\infty(\Omega)$ the space of smooth functions with compact support. Let $C_{0,\sigma}^\infty(\Omega) = \{\varphi \in C_0^\infty(\Omega) | \nabla \cdot \varphi = 0\}$. $L_\sigma^p(\mathbb{R}^3)$ denotes the closure of $C_{0,\sigma}^\infty(\Omega)$ in the norm $\|\cdot\|_p$. $\dot{H}_{0,\sigma}^{1,p}(\Omega)$ denotes the closure of $C_{0,\sigma}^\infty(\Omega)$ in the norm $\|\nabla \cdot\|_p$. We write

$$\|\mathbf{u}\|_{p,q} = \left(\int_0^T \|\mathbf{u}(t)\|_p^q dt \right)^{\frac{1}{q}}, \quad 1 \leq q < \infty,$$

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and

$$\|\mathbf{u}(t)\|_{p,\infty} = \operatorname{ess\,sup}_{0 < t < T} \|\mathbf{u}\|_p.$$

Definition 1. Let $r \geq \frac{2n}{n+2}$, $\mathbf{u}_0 \in L^2_\sigma(\Omega)$. A vector-valued function $\mathbf{u} \in L^\infty(0, T; L^2_\sigma(\Omega)) \cap L^2(0, T; \dot{H}^{1,r}_{0,\sigma}(\Omega))$ is called a weak solution to (1) if the following identity

$$(\mathbf{u}_0, \boldsymbol{\varphi}(0)) + \int_0^T \left[(\mathbf{u}, \partial_t \boldsymbol{\varphi}) - (\mathbf{u} \cdot \nabla \mathbf{u}, \boldsymbol{\varphi}) - \left(|D(\mathbf{u})|^{r-2} D(\mathbf{u}), D(\boldsymbol{\varphi}) \right) \right] dt = 0 \quad (2)$$

holds for all $\boldsymbol{\varphi} \in C_0^\infty([0, T]; C_{0,\sigma}^\infty(\Omega))$. Here, one restricts $r \geq \frac{2n}{n+2}$ to make sure that the expression $\int_0^T (\mathbf{u} \cdot \nabla \mathbf{u}, \boldsymbol{\varphi}) dt$ makes sense.

The existence of weak solutions of (1) is shown in [1,2] with the periodic boundary condition, and in [3] for the whole space. In [4], J. Wolf showed the existence of weak solutions with Dirichlet boundary condition for $r > \frac{2(n+1)}{n+2}$. Moreover, we know that a weak solution satisfies the global energy inequality

$$\|\mathbf{u}(t)\|_2^2 + 2 \int_0^t \|D(\mathbf{u})(\tau)\|_r^r d\tau \leq \|\mathbf{u}_0\|_2^2, \quad t \geq 0.$$

A natural question is to consider the possible validity of the energy equality. For Newtonian fluids, i.e. $r = 2$, the pioneering results by Prodi [5] and Lions [6] concern the validity of energy equality for a weak solution such that

$$\mathbf{u} \in L^4(0, T, L^4(\Omega)).$$

Later, Shinbrot [7] enlarged the range of exponents proving that if a weak solution belongs to

$$\mathbf{u} \in L^q(0, T; L^p(\Omega)), \quad \frac{1}{p} + \frac{1}{q} \leq \frac{1}{2}, \quad p \geq 4.$$

More results in this respect, the reader can see [8] and references therein. However, as far as I know, there is no result for the non-Newtonian fluids. In this paper, we will extend Shinbrot's result to the non-Newtonian fluids. Our main result is stated as follows.

Theorem 1. Let $r > \frac{2(n+1)}{n+2}$, r' is the Hölder conjugate of r , $\mathbf{u}_0 \in L^2_\sigma(\Omega)$, and let $\mathbf{u}$ be a weak solution of (1). If $\mathbf{u} \in L^q(0, T; L^p(\Omega))$, where $2r' \leq p \leq \frac{2r}{r-2}$ and $\frac{1}{p} + \frac{1}{q} \leq \frac{1}{r'}$, if $r > 2$; $p \geq 2r'$ and $\frac{r-1}{p} + \frac{1}{q} \leq \frac{r-1}{2}$, if $r \leq 2$. Then $\mathbf{u}$ satisfies the energy equality

$$\|\mathbf{u}(t)\|_2^2 + 2 \int_0^t \|D(\mathbf{u})(\tau)\|_r^r d\tau = \|\mathbf{u}_0\|_2^2, \quad 0 \leq t < T. \quad (3)$$

Remark 1. Here, since the existence of weak solutions to (1) is still unsolved for $r \leq \frac{2(n+1)}{n+2}$, see [4], we have to restrict $r > \frac{2(n+1)}{n+2}$.

Remark 2. It is well known that the weak solution is strong and unique for $r \geq \frac{3n+2}{n+2}$, see for example [2], thus the weak solution satisfies automatically the energy equality. It is remarkable that our result is consistent with this fact. Actually, by virtue of Gagliardo–Nirenberg inequalities and Korn's inequality, one easily verifies

$$L^\infty(0, T; L^2_\sigma(\Omega)) \cap L^2(0, T; \dot{H}^{1,r}_{0,\sigma}(\Omega)) \hookrightarrow L^{\frac{(n+2)r^2-2nr}{n}}(0, T; L^{2r'}_\sigma(\Omega)).$$

When $r \geq \frac{3n+2}{n+2}$, one can easily check that

$$\frac{1}{2r'} + \frac{n}{(n+2)r^2-2nr} \leq \frac{1}{r'},$$

which implies a weak solution of (1) must satisfy the energy equality for any $r \geq \frac{3n+2}{n+2}$.

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