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Large-time asymptotics of a fractional drift–diffusion–Poisson system via the entropy method

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The self-similar asymptotics for solutions to the drift–diffusion equation with fractional dissipation, coupled to the Poisson equation, is analyzed in the whole space. It is shown that in the subcritical and supercritical cases, the solutions converge to the fractional heat kernel with algebraic rate. The proof is based on the entropy method and leads to a decay rate in the $L^1(\mathbb{R}^d)$ norm. The technique is applied to other semilinear equations with fractional dissipation.

1. Introduction

In this paper, we investigate the large-time behavior of solutions to a drift–diffusion equation with fractional diffusion, coupled self-consistently to the Poisson equation. Such models describe the evolution of particles in a fluid under the influence of an acceleration field. The particle density $\rho(x,t)$ and potential $\psi(x,t)$ satisfy the equations

$$\partial_t\rho + (-\Delta)^{\theta/2}\rho = \operatorname{div}(\rho\nabla\psi), \quad -\Delta\psi = \rho \quad \text{in } \mathbb{R}^d,\ t>0, \tag{1}$$

with initial condition

$$\rho(\cdot,0) = \rho_0 \quad \text{in } \mathbb{R}^d. \tag{2}$$

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The fractional Laplacian $(-\Delta)^{\theta/2}$ is defined by $(-\Delta)^{\theta/2}\rho = \mathcal{F}^{-1}[|\xi|^\theta \mathcal{F}[\rho]]$, where $\mathcal{F}$ is the Fourier transform, $\mathcal{F}^{-1}$ its inverse, and $\theta > 0$. When $\theta = 2$, we recover the standard drift–diffusion–Poisson system arising in semiconductor theory and plasma physics [21]. Drift–diffusion-type equations with $\theta < 2$ were proposed to describe chemotaxis of biological cells whose behavior is not governed by Brownian motion [16]. For given acceleration field $\nabla\psi$ and $1 < \theta < 2$, the first equation in (1) was derived from the Boltzmann equation by Aceves-Sanchez and Mellet [1], based on the moment method of Mellet [28]. For the convenience of the reader, we present a formal derivation of the coupled system in Appendix B. Note, however, that we consider Eqs. (1) in the range $0 < \theta \le 2$.

The aim of this paper is to compute the decay rate of the solution to (1) to self-similarity in the $L^1(\mathbb{R}^d)$ norm using the entropy method. In previous works [7,26,43], the self-similar asymptotics of various model variations were shown in $L^p(\mathbb{R}^d)$ norms but the decay rate is zero when $p = 1$. If additionally the first moment exists, i.e. if $|x|\rho_0 \in L^1(\mathbb{R}^d)$, the decay rate of the self-similar asymptotics in the $L^1(\mathbb{R}^d)$ norm is $1/\theta$, which is optimal [31, Lemma 5.1]. The entropy method provides an alternative way to analyze the self-similar asymptotics in the $L^1(\mathbb{R}^d)$ norm. Moreover, its strength is its robustness, i.e., the method can be easily applied to other semilinear equations with fractional dissipation. We give two examples in Section 4.

Before stating our main result and the key ideas of the technique, we review the state of the art for drift–diffusion equations. The *global existence of solutions* to the drift–diffusion–Poisson system was shown for $\theta = 2$ in [25], for the subcritical case $1 < \theta \le 2$ in [26,31], and for the supercritical case $0 < \theta < 1$ in [26,35]. For suitable initial data, the solution to (1) satisfies

$$\rho \in C^0([0,\infty); L^1(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)), \quad \rho(t) \ge 0 \quad \text{in } \mathbb{R}^d,\ t > 0, \tag{3}$$

$$\|\rho(t)\|_{L^1(\mathbb{R}^d)} = \|\rho_0\|_{L^1(\mathbb{R}^d)}, \quad \|\rho(t)\|_{L^p(\mathbb{R}^d)} \le C(1+t)^{-\frac{d}{\theta}(1-\frac{1}{p})}, \quad 1 \le p \le \infty; \tag{4}$$

see [31, Theorem 1.1] for $1 < \theta \le 2$, [41, Theorem 1] for $\theta = 1$ and $d \ge 3$, and [26, Theorem 1.7] for $0 < \theta < 1$ and $d = 2$. The existence result of [31] for the bipolar drift–diffusion system was extended by Granero-Belinchón [18] by allowing for different fractional exponents. The existence of solutions in Besov spaces was proved for $1 < \theta < 2d$ and $d \ge 2$ in [44], and for $0 < \theta \le 1$ and $d \ge 3$ in [35].

The self-similar asymptotics of the *fractional heat equation* was studied by Vazquez [37, Theorem 3.2], showing that if $0 < \theta < 2$ and the initial datum satisfies $(1 + |x|)\rho_0 \in L^1(\mathbb{R}^d)$, we have $\|\rho(t) - MG_\theta(t)\|_{L^1(\mathbb{R}^d)} \le Ct^{-1/\theta}$ with optimal rate, where $M = \int_{\mathbb{R}^d} \rho_0\, dx$ is the initial mass and G_θ is the fundamental solution to the fractional heat equation (see Section 2.1). Exploiting the self-similar structure, this proves the exponential decay with rate $1/\theta$ to the fractional Fokker–Planck equation with quadratic potential. The exponential decay in L^1 spaces with weight $1 + |x|^k$ and $k < \theta$ was proved by Tristani [36]. The fractional Laplacian can be replaced by more general Lévy operators, and the large-time asymptotics of so-called Lévy–Fokker–Planck equations were investigated by Biler and Karch [6] as well as Gentil and Imbert [17].

The large-time behavior of solutions to *drift–diffusion–Poisson systems* with $d = 2$ and $\theta = 2$ was studied by Nagai [30], showing the decay of the solutions to zero. A similar result for $0 < \theta < 2$ was proven by Li et al. [26]. The self-similar asymptotics in $L^p(\mathbb{R}^d)$ with $1 \le p \le \infty$ was shown in [23] for $\theta = 2$ and in [31] for $1 < \theta < 2$. In the latter reference, also the decay of the first-order asymptotic expansion of the solutions was computed. Higher-order expansions were studied for $1 < \theta \le 2$ in [40], for $0 < \theta \le 1$ in [43], and for the critical case $\theta = 1$ in [42]. However, in most of these references, the decay rate for $p = 1$ is zero.

The exponential decay in the relative entropy for solutions to Lévy–Fokker–Planck equations was proved in [6,17]. Via the Csiszár–Kullback inequality (see, e.g., [3]), this implies decay in the $L^1(\mathbb{R}^d)$ norm. In fact, we are using the techniques of [17], combined with tools from harmonic analysis and semigroup theory, to achieve self-similar decay of solutions to (1).