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Non-radial solutions for some semilinear elliptic equations on the disk

Gianni Arioli[a,*], Hans Koch[b]

[a] Department of Mathematics, Politecnico di Milano, Piazza Leonardo da Vinci 32, 20133 Milano, Italy
[b] Department of Mathematics, University of Texas at Austin, Austin, TX 78712, United States

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ABSTRACT

Starting with approximate solutions of the equation $-\Delta u = wu^3$ on the disk, with zero boundary conditions, we prove that there exist true solutions nearby. One of the challenges here lies in the fact that we need simultaneous and accurate control of both the (inverse) Dirichlet Laplacean and nonlinearities. We achieve this with the aid of a computer, using a Banach algebra of real analytic functions, based on Zernike polynomials. Besides proving existence, and symmetry properties, we also determine the Morse index of the solutions.

1. Introduction

In this paper we consider semilinear elliptic equations of the form

$$-\Delta u = wf'(u)\,, \qquad u\big|_{\partial\Omega} = 0\,, \tag{1.1}$$

where Ω is the unit disk in $\mathbb{R}^2$, w is a nonnegative function on Ω, and f' is the derivative of a regular function f on $\mathbb{R}$. In the cases considered here, w is always radial (invariant under rotations) and $f'(u) = u^3$. But it will be clear from our description that the same methods work for other choices of w and f. In fact, similar techniques should apply to other types of equations, and to other radially symmetric domains in $\mathbb{R}^2$ and $\mathbb{R}^3$.

Before giving more details, let us state a result that will help to set the stage.

Theorem 1.1. *There exists a positive radial polynomial w on Ω, such that Eq. (1.1) with $f'(u) = u^3$ admits a real analytic solution $u = u_w$ that has Morse index 2, with the property that $|u_w|$ is not invariant under any nontrivial rotation.*

The weight function w and the solution u_w are shown in Fig. 1. A precise definition of w is given in [1]. We note that u_w is symmetric under a reflection. This is one symmetry that solutions cannot avoid [17]. Our goal was to find an index-2 solution that has no other symmetries.

* Corresponding author.
E-mail address: gianni.arioli@polimi.it (G. Arioli).

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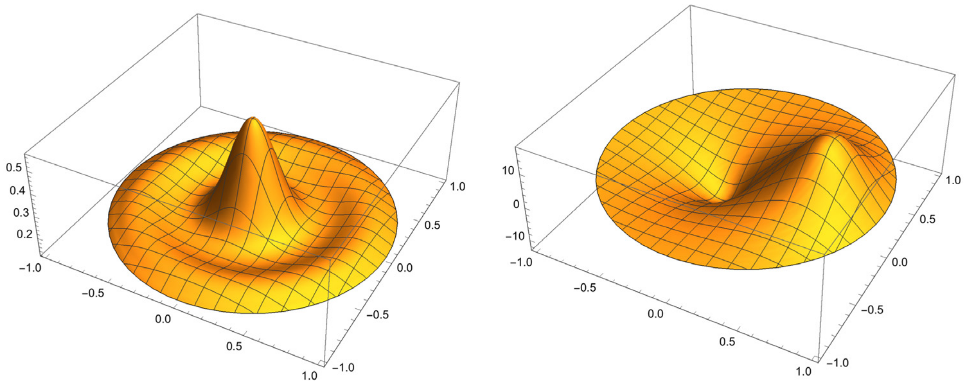

Fig. 1. The weight function w and solution u_w described in Theorem 1.1.

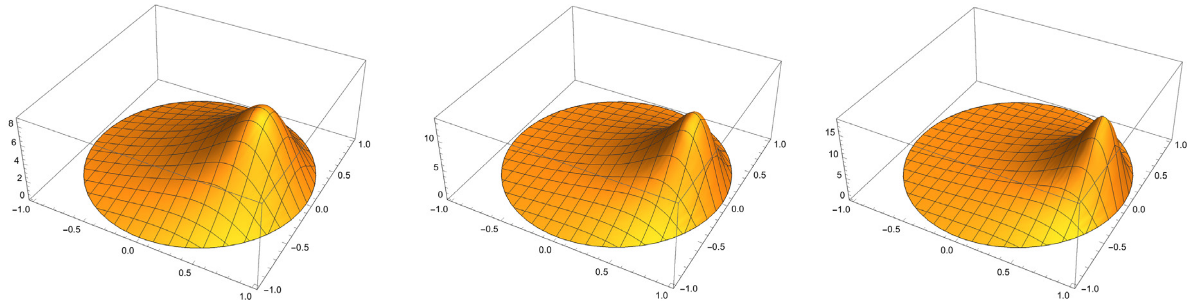

Fig. 2. The solutions u_2, u_4, and u_6 described in Theorem 1.2.

Concerning the Morse index, recall that solutions of Eq. (1.1) are critical points of the functional J on $\mathrm{H}_0^1(\Omega)$,

$$J(u) = \int_\Omega \left[\frac{1}{2}|\nabla u|^2 - wf(u)\right] dxdy\,, \tag{1.2}$$

assuming that f satisfies some growth and regularity conditions. The Morse index of a critical point u is the number of descending directions of J at u.

One of the difficulties with proving Theorem 1.1 is that Ω is a disk. For a square domain, an analogous result was proved in [2]. And for the disk, it is possible [3] to obtain an accurate numerical "solution" that looks as shown in Fig. 1. But we have hitherto been unable to prove that there exists a true solution nearby.

Before describing our approach in more detail, let us state two other results that can be proved in a similar way. The first results concern again "minimally symmetric solutions to a highly symmetric problem". While the weight w in Theorem 1.1 had to be chosen carefully to obtain a minimally symmetric solution of index 2, a standard Hénon weight $w(r,\vartheta) = r^\alpha$ suffices in the index-1 case. Here, and in what follows, (r,ϑ) denote the standard polar coordinates on Ω.

Theorem 1.2. *For $\alpha = 2, 4, 6$, Eq. (1.1), with $w = r^\alpha$ and $f'(u) = u^3$, admits a real analytic solution $u = u_\alpha > 0$. This solution has Morse index 1 and is not invariant under any nontrivial rotation.*

The solutions u_2, u_4, and u_6 are shown in Fig. 2.

The same result, but without the statement about the lack of symmetry, is easy to prove: minimizing J on the Nehari manifold $\mathfrak{N} = \{u \in \mathrm{H}_0^1(\Omega) : DJ(u)u = 0\,, u \neq 0\}$ shows that index-1 solutions exist and