Nonlinear Analysis 179 (2019) 354–382

Contents lists available at ScienceDirect

Nonlinear Analysis

www.elsevier.com/locate/na

Stable maximal hypersurfaces in Lorentzian spacetimes

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A R T I C L E I N F O

Article history:
Received 29 August 2018
Accepted 12 September 2018
Communicated by Enzo Mitidieri

MSC:
53C42
53C50
83C99
53C80

Keywords:
Stable maximal hypersurface
k-stability
Spacetime of constant sectional curvature
Generalized Robertson–Walker spacetime

A B S T R A C T

We study the geometry of stable maximal hypersurfaces in a variety of spacetimes satisfying various physically relevant curvature assumptions, for instance the Timelike Convergence Condition (TCC). We characterize stability when the target space has constant sectional curvature as well as give sufficient conditions on the geometry of the ambient spacetime (e.g., the validity of TCC) to ensure stability. Some rigidity results and height estimates are also proven in GRW spacetimes. In the last part of the paper we consider k-stability of spacelike hypersurfaces, a concept related to mean curvatures of higher orders.

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1. Introduction

In the last decades, maximal hypersurfaces in spacetimes have attracted a great deal of mathematical and physical interest. The importance of this family of spacelike hypersurfaces in General Relativity is well-known and a summary of several reasons justifying this opinion can be found, for instance, in [27]. Among them, we emphasize the key role they play in the study of the Cauchy problem [16,25] as well as their importance in the proof of the positivity of the gravitational mass [34]. Furthermore, maximal hypersurfaces describe, in some relevant cases, the transition between the expanding and contracting phases of a relativistic universe. Finally, the existence of constant mean curvature (and in particular maximal) hypersurfaces is useful in the study of the structure of singularities in the space of solutions of the Einstein equations [6]. At last, we should also mention their use in numerical relativity for integrating forward in time [24].

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https://doi.org/10.1016/j.na.2018.09.009
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From a mathematical point of view, maximal hypersurfaces in a spacetime $\overline{M}$ are (locally) critical points for a natural variational problem, namely, that of the area functional (see, for instance, [9]) and their study is helpful for understanding the structure of $\overline{M}$ [7]. In particular, for some asymptotically flat spacetimes, maximal hypersurfaces produce a foliation of the spacetime, defining a time function [11]. Classical papers dealing with uniqueness of maximal hypersurfaces are, for instance, [11,15], although a previous relevant result in this direction was the proof given by Cheng and Yau [14] of the Bernstein–Calabi conjecture [12]: spacelike affine hyperplanes are the only complete maximal hypersurfaces in the $(m+1)$-dimensional Lorentz–Minkowski spacetime. Nishikawa [28] extended their result by proving that any complete maximal hypersurface immersed in a spacetime $\overline{M}$ is totally geodesic when $\overline{M}$ belongs to a family of locally symmetric Lorentzian manifolds that includes spacetimes of nonnegative constant curvature. Ishihara [23] showed that this property is not shared by spacetimes of negative constant curvature by exhibiting an example of a complete maximal hypersurface with constant nonzero norm of the shape operator in the $(m+1)$-dimensional anti-de Sitter spacetime of curvature -1. In Theorem 7 we prove a slight generalization of Nishikawa's result by proving an upper bound on the norm of the shape operator first obtained by Ishihara in the case of ambient spacetimes of constant curvature. More recently, new uniqueness results for maximal hypersurfaces have been found in a large variety of spacetimes by means of different techniques [5,31,32].

In this paper we will focus on a particular family of maximal hypersurfaces, namely, stable maximal hypersurfaces, that is, critical points of the volume functional for compactly supported normal variations with non-positive second variation. A mild condition on the curvature of the ambient spacetime is enough to ensure stability of maximal hypersurfaces.

Theorem A. *Let $\overline{M}$ be a spacetime with nonnegative Ricci curvature on timelike vectors. If $\psi : M \longrightarrow \overline{M}$ is a (not necessarily complete) oriented maximal hypersurface, then ψ is stable. If M is also compact, then ψ is totally geodesic.*

Note that in an oriented spacetime $\overline{M}$, the time orientation of $\overline{M}$ ensures that every spacelike hypersurface is oriented. In General Relativity, a spacetime with nonnegative Ricci curvature on timelike vectors is said to obey the Timelike Convergent Condition (TCC). It is usually argued that the TCC is the mathematical way to express that gravity, on average, attracts (see [29]). Theorem A generalizes Corollary 5.6 of [5] and Theorem 1 of [31], where the authors show that compact maximal hypersurfaces in a spacetime $\overline{M}$ obeying the TCC are totally geodesic by also assuming the existence of certain infinitesimal symmetries in $\overline{M}$. As a corollary, we also have an alternative proof of Theorem 4.1 of [11], a uniqueness result for vacuum spacetimes.

Corollary B. *Let M be a compact maximal hypersurface in a spacetime that obeys the Einstein vacuum equations without cosmological constant. Then, M is totally geodesic.*

If a maximal hypersurface $\psi : M \to \overline{M}$ is unstable, then there exist spacelike hypersurfaces of larger volume in $\overline{M}$ nearby ψ. This happens, for instance, for the equator of de Sitter spacetime, which is a saddle point for the volume functional. In fact, we have the following

Theorem C. *Let $\overline{M}$ be an $(m+1)$-dimensional spacetime of constant curvature $\overline{\kappa}$ and let $\psi : M \longrightarrow \overline{M}$ be a complete oriented maximal hypersurface.*

(i) If $\overline{\kappa} > 0$ then M is compact and the immersion ψ is totally geodesic and unstable.
(ii) If $\overline{\kappa} = 0$ then ψ is totally geodesic and stable.
(iii) If $\overline{\kappa} < 0$ then ψ is stable and the shape operator A and the scalar curvature S of M satisfy

$$\mathrm{trace}(A^2) \leq -m\overline{\kappa}, \qquad \mathrm{S} \leq (m-2)m\overline{\kappa}.$$

If M is also compact, then ψ is totally geodesic.

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