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Waves without the wave equation: Examples from nonlinear acoustics

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ABSTRACT

The traditional wave equation is mostly, if not always, obtained from a system of first order partial differential equations augmented by constitutive relations. These are often nonlinear and linearizations are forcibly applied.

In a nonlinear system of first order partial differential equations the criterion for hyperbolicity, necessary for the description of wave phenomena, involves the solution. It is therefore possible that solutions may evolve in such a way that hyperbolicity is *challenged* in the sense that the system comes close to not being hyperbolic. We use the recently introduced formulation for nonlinear acoustic disturbances to illustrate. When hyperbolicity deteriorates, standard numerical methods and the heuristics surrounding wave motion may be compromised. To overcome such difficulties we introduce the notion of *inverse characteristic* which, at least in the examples, reduces numerical calculations to elementary techniques and clarifies intuition. Analysis of inverse characteristics leads to two systems of ordinary differential equations that have *time-like* trajectories and *space-varying* associated curves. Time-like trajectories give rise to an alternative measure of time in terms of which space-like trajectories are easier to analyze. Space-varying curves enable the analysis of shock phenomena in a direct way. We give conditions under which an initially mild challenge of hyperbolicity, represented by pressure, develops into a severe challenge. Under these conditions violent velocity shocks develop from an initially undisturbed state.

1. Introduction

In 1747 D'Alembert unveiled the wave equation in one space dimension and the general form of its solutions. To some mathematicians and those who apply mathematics, this was the beginning of a life-long romance. If one carefully scrutinizes the various derivations a number of critical points turn up. For instance, in an elastic bar longitudinal disturbances cannot be one-dimensional and paradoxical consequences are not hard to come by. After all, the bar would become thinner when stretched and thicker where it is compressed!

The equations for wave-like disturbances are, within the context of continuum mechanics, based on balance of linear momentum, conservation of mass and constitutive relations. These usually manifest in first order partial differential equations. From there ruthless procedures of elimination and linearization lead to the dearly beloved and greatly admired wave equation.

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It was in 1958 that K. O. Friedrichs, in his study of positive linear systems partial differential equations, pointed out that the wave equation could be derived from a system of first order equations, thus taking a step back towards the underlying physical principles.

In a recent paper, Sauer (2010), the equations of nonlinear acoustics were re-formulated in terms of the so-called Lagrangian description of fluid motion (introduced by Euler). In Sauer (2011) the formulation is discussed in a more succinct manner. The use of a coordinate system that follows every 'point' in a reference configuration was used to accommodate the principle of a materially closed system. Balance of momentum and conservation of mass combined into a system of first order equations. The equation of state for gas under isentropic conditions lead to the constitutive equation. Further analysis revealed that the first order system of governing partial differential equations is also subject to an inequality constraint derived from physical considerations.

For situations with constant density in the reference configuration the equations can be scaled to dimensionless form and if the motion is one dimensional in space, the differential equations are

$$\left.\begin{aligned} v_t(t,x) + p_x(t,x) &= 0; \\ p_t(t,x) + [1+p(t,x)]^2 v_x(t,x) &= 0. \end{aligned}\right\} \tag{1}$$

Here $v(t, x)$ is the dimensionless velocity of "particle" $x \in \mathbb{R}$ in the reference configuration at time $t > 0$ and $p(t, x)$ the dimensionless pressure experienced by it. Velocity is scaled according to "thermostatic sound speed" which then corresponds to $v = 1$. It is of interest to note that this system of equations is not invariant under interchange of the variables v and p. For systems that lead to the wave equation this invariance is essential. Brute force elimination of v in (1), with many liberties taken, leads to the nonlinear wave equation

$$p_{tt} - \frac{2p_t^2}{1+p} - [1+p]^2 p_{xx} = 0.$$

It may be better to stay with (1) which is close to the original derivation.

The constraint, expressed in the form

$$1 + p(t,x) > 0, \tag{2}$$

needs elucidation. If $J(x, t)$ is the Jacobian of the motion $X(t) = \Psi(x,t)$ it is shown in Sauer (2010) that the principle of conservation of mass can be expressed in the form $\rho(x) = J(x,t)\sigma(X,t)$ with $\rho(x)$ the mass-density in the reference configuration and $\sigma(X, t)$ the mass density at time t. Hence, for densities to be positive, it is a physical requirement that $J(x, t) > 0$. It is also shown that a *Boyle–Mariotte relation* holds in the form $[1 + p(t,x)]J(x,t) = 1$. The constraint is derived from the positivity of J. Mathematically it ensures that the system (1) is hyperbolic. In fact, the crucial eigenvalues that determine hyperbolicity are $1 + p$ and $-(1 + p)$. Along with this are two eigenvectors of the form $([1 + p], 1)$ and $(-[1 + p], 1)$ which are linearly independent. Hyperbolicity of a system is essential for it to exhibit wave-like propagation of disturbances. It is challenged when $1 + p(t,x)$ approaches zero. Then the corresponding eigenvectors are nearly the same. This can cause computational difficulties. Even worse, since hyperbolicity depends on the solution, a challenge may develop as time goes on. In turn the wave-like character of solutions could come close to being lost.

An example in Sauer (2010) illustrates the strange behaviour exhibited by the system when the initial pressure challenges the constraint for large x. The case in question is where initially $v(0,x) = 0$ and *This is okay in PDF – version*. Rough, limited numerical computations of characteristics revealed the curious behaviour that the (bell-shaped) initial pressure profile decays towards the limit enforced by the constraint, so that hyperbolicity becomes severely challenged. On the other hand the gas, initially at rest, becomes increasingly unsettled as time goes on. Also, activity appeared to be confined to an interval outside of which the Gaussian curve does not contribute significantly to the occurrence of events. In the present paper we develop tools to investigate this observation in a precise manner for a wide class of initial pressures. The focus will mostly be on cases where $1 + p(0,x) \to 0$ as $x \to \infty$ which means that the constraint is initially challenged "far away." The case $x \to -\infty$ is similar.

To deal with the constraint, the variable q is introduced as follows:

$$1 + p(t,x) = \exp\{q(t,x)\}. \tag{3}$$

This transforms the system (1) into

$$\left.\begin{aligned} v_t(t,x) + \exp\{q(t,x)\}q_x(t,x) &= 0; \\ q_t(t,x) + \exp\{q(t,x)\}v_x(t,x) &= 0. \end{aligned}\right\} \tag{4}$$

The quasi-linear system (4) is symmetric hyperbolic (see *e.g.* Jeffrey, 1976, Lax (Lax, 1973)), and can be transformed to canonical form by the substitutions

$$\left.\begin{aligned} u_1(t,x) &= \tfrac{1}{2}[v(t,x) + q(t,x)]; \\ u_2(t,x) &= \tfrac{1}{2}[v(t,x) - q(t,x)], \end{aligned}\right\} \tag{5}$$