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Can derivative determine the dynamics of fractional-order chaotic system?

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ABSTRACT

Dynamics, control and applications of fractional-order systems are important issues in nonlinear science, and have received increasing interests in recent years. However, most of the existing studies are carried out based on the constant order fractional (COF) nonlinear systems. Moreover, there are few publications on bifurcation, chaos, complexity of variable order fractional (VOF) nonlinear systems. In this paper, variable fractional derivative orders such as periodical signal, noise signals are introduced into the fractional-order Simplified Lorenz chaotic system. Numerical solution, LEs, complexity measuring algorithms are proposed, and how the dynamics can be changed by the variable orders is discussed in detail. The results indicate that the dynamics of the VOF can be controlled by the designed variable orders and the VOF Simplified Lorenz system has higher complexity than its COF counterpart. The results demonstrate the effectiveness and advantages of the proposed method and the engineering application worth of the VOF systems.

1. Introduction

Fractional-order calculus was proposed by Leibniz and L'Hôspital in 1695, but it is only concerned in mathematics for three centuries. Until recently it has received a great deal of attention in nonlinear science. Specifically, it has received extensive attention on the differen fields such as anomalous diffusion [1], imaging science [2] and quantum Physics [3] because memory properties have been widely found in these complex systems.

In the last two decades, dynamics, stability, control, synchronization and applications of fractional-order chaotic systems have attracted many researchers' interests. Li and Chen [4] investigated chaotic behaviors of the fractional-order Chen system, and found that the minimum order for chaos is 2.1. Afterwards, bifurcation and chaos of a Simplified Lorenz system are analyzed by Sun et al. [5] and Wang et al. [6] by employing predictor-corrector algorithm and Adomian decomposition method, respectively. Currently, chaos is also fond in the fractional-order memristive system [7] and fractional complex dynamical networks [8]. Fractional stability theory was proposed in Ref. [9], and it is widely used to design different controllers of fractional systems [10–12]. Meanwhile different synchronization schemes such as active synchronization [13], fuzzy synchronization [14], projective synchronization [15] for the fractional-order systems have been proposed by employing the fractional stability theory. Moreover, Teng et al. [16] designed the analog circuit of fractional chaotic system while He et al. [17] realized the fractional-order Lorenz hyperchaotic system in the DSP digital circuit. Finally, it is important to note that fractional-order chaotic systems can be used in the in many engineering fields such as image encryption [18], voice encryption [19] and secure communication [20].

In fact, the most works about fractional systems are carried out in the systems where orders are constant, namely, in the constant-order fractional (COF) systems. In fact, derivative orders in fractional-order systems which means memory effect usually varies with nonlinear external disturbance in reality, and these physical phenomena can be described better by introducing the VOF derivative. For example, the trapped Brownian particle in viscoelastic shear flows was investigated in Ref. [21]., where the influence of a fluctuating environment is modeled by a multiplicative white noise (fluctuations of the stiffness of the trapping potential) and by an additive internal fractional Gaussian noise. Moreover, the traditional COF derivative can be regarded as the special case of the VOF derivative, thus investigation of VOF derivative is

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of significance much more. Currently, there is an increasing interest on analysis of variable-order nonlinear systems [22–24]. For instance, Sun et al. [22] made a comparison between constant-order and variable-order derivatives and two types of variable-order fractional derivatives in characterizing the memory property of systems are discussed. Atangana [24] found chaos in the nonlinear coupled reaction-diffusion system with four-dimensional space. Meanwhile, there are also a few studies directly analyzing variable-order fractional (VOF) chaotic systems, but they are mainly focusing on synchronization of those VOF chaotic systems [25–27]. For example, Xu and He [25] analyzed finite-time anti-synchronization of two VOF chaotic systems based on the Mittag-Leffler stable theory. All in all, to our best knowledge, variable order fractional derivative is at its early stages of research, and there are still many studies about VOF derivative and VOF nonlinear systems needed to be carried out. Besides, there is few work specially investigating numerical solution, bifurcation and chaos of the VOF chaotic systems.

Inspired by the above discussions, we raise a question: can derivative determine dynamics of a fractional-order chaotic system. To answer this question, the rest of the paper is organized as follows. In Section 2, the definition of VOF Caputo calculus and its discrete form are given firstly. Then a semi-analytic solution algorithm and the Lyapunov exponents (LEs) calculation algorithm are proposed and FuzzyEn algorithm for complexity measuring is presented. In Section 3, phase portraits, dynamics and complexity of the VOF Simplified Lorenz system are analyzed under different kinds of variable orders. In Section 4, how dynamics of the VOF Simplified Lorenz system is controlled by the variable orders and the control mechanism are discussed. Finally, the conclusion in Section 5 is presented.

2. Variable-order chaotic system and its numerical analysis methods

2.1. Definitions

Firstly, we introduce the VOF Caputo derivative definition which is denoted as Definition 1.

definition 1 ([28]). The variable-order derivative Caputo definition is given by

$$D_{t_0}^{q(t)}x(t) = \frac{1}{\Gamma(1-q(t))}\int_{t_0}^{t}(t-\tau)^{-q(t)}f'(\tau)d\tau, \tag{1}$$

in which $0<q(t)\le 1$ is a function of time t, and $\Gamma(\cdot)$ is the gamma function.

Since the VOF calculus can be used in the nonlinear chaotic system and there is no exact analytical solution of the system, numerical simulation is alternative way for the analysis of the system. To get the numerical solution, the VOF Caputo definition is discretized, and it is given in Definition 2.

definition 2. The discrete VOF Caputo derivative is defined as

$$D_{t_0}^{q(t)}x(t) = \begin{cases} D_{t_0}^{q(t_0)}x(t), t_0 \le t < t_1 \\ D_{t_1}^{q(t_1)}x(t), t_1 \le t < t_2 \\ \vdots \\ D_{t_{n-1}}^{q(t_{n-1})}x(t), t_{n-1} \le t < t_n \\ \vdots \end{cases}, \tag{2}$$

where $t_n - t_{n-1} = h(n=0,1,2,\ldots)$ is the width of time interval.

Remark 1. When $h = t_n - t_{n-1} \to 0$, the discrete VOF Caputo derivative is equivalent to that defined in Definition 1. Thus, in the time interval $[t_{n-1},\ t_n]$, the order does not change, and it is the normal Caputo definition.

Remark 2. In the real calculation, we need to choose a proper value for h. A proper h should be much smaller than the frequency of curve $q(t)$.

Remark 3. The range of order function $q(t)$ should be defined within (0,1]. Meanwhile, those order functions are mainly designed by researchers. For instance, in Ref. [22], $q(t) = 1.0-0.04t, t\in[0, 5.0]$ and $q(t) = \begin{cases}0.8, t\in[0,2.0]\\ 1.0, t\in(2.0,5.0]\end{cases}$, while in Ref. [26], $q(t) = 0.09\sin(1/t)$ and $q(t) = 0.08\sin(1/t)+0.92$. In this manuscript, variable order functions can be continuous equations as defined above, can be discrete values or even noise signals.

definition 3. The VOF system with Caputo sense is defined as

$$D_{t_0}^{q(t)}\mathbf{x}(t) = f(\mathbf{x}(t)), \tag{3}$$

where $\mathbf{x}(t) = [x_1(t), x_2(t), \ldots, x_n(t)]^T$.

definition 4 ([28]). Suppose that $q\in(0,\ 1]$ is a constant, $t\in[t_{n-1},\ t_n)$, and $J_{t_{n-1}}^{q}$ is the inverse operation operator of $D_{t_{n-1}}^{q}$, which is given by

$$J_{t_{n-1}}^{q}x(t) = \frac{1}{\Gamma(q)}\int_{t_{n-1}}^{t}(t-\tau)^{q-1}x(\tau)d\tau. \tag{4}$$

Lemma 1 ([28,29]). *Assume that* x: $[t_0,\ \infty)\to R$, $t\in[t_{n-1},\ t_n)$ *and* $q(t_{n-1})\in(0,1]$, *then for* $t\in[t_{n-1},\ t_n)$, *we have*

$$J_{t_{n-1}}^{q(t_{n-1})}D_{t_{n-1}}^{q(t_{n-1})}x(t) = x(t) - x(t_{n-1}). \tag{5}$$

Lemma 2 ([28,29]). *When* $t\in[t_{n-1}, t_n]$, $0<q(t_{n-1})$, $r\le 1$ *and* C *is a given real constant, it has following results*

$$\begin{cases} J_{t_{n-1}}^{q(t_{n-1})}D_{t_{n-1}}^{q(t_{n-1})}x(t) = x(t)-x(t_{n-1}) \\ J_{t_{n-1}}^{q(t_{n-1})}J_{t_{n-1}}^{r}x(t) = J_{t_{n-1}}^{q(t_{n-1})+r}x(t) \\ J_{t_{n-1}}^{q(t_{n-1})}C = \frac{C}{\Gamma(q(t_{n-1})+1)}(t-t_{n-1})^{q(t_{n-1})} \end{cases}. \tag{6}$$

2.2. Design of VOF adomian decomposition method

Adomian decomposition method (ADM) [30] can get the analytical solution of fractional-order chaotic systems. However, there are no reports about ADM based numerical solution of VOF chaotic systems. Based on Definition 2, we can overcome the difficulty by dividing the computation interval $[t_0,\ t]$ into N subintervals $[t_n, t_{n+1}]$, n=0, 1,... , N-1 of equal step size $h = t_{n+1}-t_n = \frac{t-t_0}{N}$. The fractional-order in each subinterval is $q(t_n)$, where $t_n = (n-1)h$. To obtain a discrete solution from fractional-order chaotic system, the VOF Adomian decomposition method (VOFADM) is designed.

Firstly, we separate the system $f(\mathbf{x}(t))$ into three terms

$$D_{t_0}^{q(t)}\mathbf{x}(t) = L\mathbf{x}(t) + N\mathbf{x}(t) + \mathbf{g}, \tag{7}$$

where $L\mathbf{x}(t)$, $N\mathbf{x}(t)$ and $\mathbf{g}$ are the linear, nonlinear and constant terms of the fractional differential equations, respectively. Without loss of generality, the operator $J_{t_0}^{q}$ is applied to both side of Lemma 1 at the subinterval $[t_n, t_{n+1}]$, and the following equation is obtained [29,30]

$$\mathbf{x}(t_{n+1}) = J_{t_n}^{q(t_n)}L\mathbf{x}(t_n) + J_{t_n}^{q(t_n)}N\mathbf{x}(t_n) + J_{t_n}^{q(t_n)}\mathbf{g} + \mathbf{x}(t_n). \tag{8}$$

According to ADM, the decomposition solution can also be denoted as

$$\mathbf{x}(t_{n+1}) = \sum_{i=0}^{\infty}\mathbf{x}^i(t_{n+1}) = F(\mathbf{x}(t_n)). \tag{9}$$