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Laplacian spectrum and coherence analysis of weighted hypercube network

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ABSTRACT

Hypercube network is one of the most important and attractive network topologies so far. In this paper, we consider the scaling for first- and second-order network coherence on the hypercube network controlled by a weight factor. Our objective is to quantify the robustness of algorithms to stochastic disturbances at the nodes by using a quantity called network coherence which can be characterized as Laplacian spectrum. Network coherence can capture how well a network maintains its formation in the face of stochastic external disturbances. Firstly, we deduce the recursive relationships of its eigenvalues at two successive generations of Laplacian matrix. Then, we obtain the Laplacian spectrum of Laplacian matrix. Finally, we calculate the first- and second-order network coherence quantified as the sum and square sum of reciprocals of all nonzero Laplacian eigenvalues by using Squeeze Theorem. The obtained results show that the network coherence depends on generation number and weight factor. Meanwhile, the scalings of the first- and second-order network coherence of weighted hypercube decrease with the increasing of weight factor r, when $0 < r < 1$.

1. Introduction

In the past decade, the study of networks associated with complex systems has received the attentions of researchers from different scientific fields, such as physics, mathematics. Previous research mainly focus on binary network, and there are few research on weighted network. Weighted networks are extension of networks or graphs [1–3], in which the edge between nodes i and j is associated with a variable w_{ij}, called the weight. Practical realizations of weights in real networks range from the number of passengers traveling yearly between two airports in airport networks [4], to the traffic measure in packets per unit time between routers in the Internet [5] or the intensity of predator-prey interactions in ecosystems [6]. Hence it is necessary for a modeling approach that can capture the effects of weighted characteristics on complex dynamics. One of the innovation in this paper is the use of the weight factor, we believe that this innovation will open new perspectives for some studies based on binary networks.

Distributed consensus algorithms are important tools in the domains of multiagent systems and the vehicle platooning problem [7–9] as a means by which agents can reach and maintain agreement on quantities such as heading, velocity, and inter-vehicle spacing using only local communication. In these settings, it is also important to consider how robust these algorithms are to external disturbances in addition to verifying the correctness of distributed consensus algorithms. For the dynamical processes on complex network, there are some specific applications, such as the study of SIS model with an infective vector on annealed networks and the study of SIR dynamics on quenched networks [10,11]. Several recent works have studied the robustness of distributed consensus algorithms for systems with first- and second-order dynamics according to an H_2 norm. This norm is a quantification of the network coherence and can capture how well a network maintain its formation when facing stochastic external disturbances. For systems with first-order dynamics, it has been shown that the H_2 norm can be characterized by the trace of the pseudo-inverse of the Laplacian matrix [12–15]. This value has significant meaning not just in consensus systems, but in electrical networks [16,17], random walks [18], and molecular connectivity [19]. For the second-order case, the H_2 norm is also determined by the spectrum of the Laplacian. For example, Bamieh et al. proposed an asymptotic analysis of network coherence for the first- and second-order consensus algorithms in torus and lattice networks according to number of nodes and network dimension [15] and then Patterson and Bamieh gave the first- and second-order consensus algorithms in networks with stochastic disturbances [20].

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The first- and second-order network coherence quantified as the sum and square sum of reciprocals of all nonzero Laplacian eigenvalues. Our recent works [21,22] gave scalings for the H_2 norm of first- and second-order consensus algorithms in weighted small-world networks and weighted iteration trees. These two works have some similarities, such as the obtained exact expressions of the network coherence by calculating. However, the analytical determination of Laplacian spectrum and the calculation about the sum of reciprocals of all nonzero Laplacian eigenvalues are tedious, even the numerical calculation of the Laplacian eigenvalues is limited by the order of the graph and non-practical for large networks [23–25]. In this paper, we introduce a weighted hypercube based on binary hypercube and obtained the scalings of first- and second-order network coherence. The leading term of network coherence can be obtained by using Squeeze Theorem, which bypasses quite a lot of sophistical mathematical analysis. That is to say, this method greatly simplifies the process of calculation which is another innovation point in this paper.

The organization of this paper is as follows. In Section 2, we introduce the definition of network coherence. In Section 3, we give the model of the weighted hypercube network. In Section 4, we obtain the relationship of Laplacian eigenvalues at two successive generations and compute the network coherence of the weighted hypercube. Finally, we obtain the scalings of the first- and second-order network coherence. In Section 5, we draw a conclusion.

2. Network coherence

We consider linear first-order consensus over a network modeled by a weighted network G^w with N nodes and E edges. Let W and S be the weighted adjacency matrix and weighted diagonal degree matrix of G^w. The Laplacian matrix of the network G^w is denoted by L and is defined as $L = S - W$. Our objective is to investigate consensus dynamics in a linear dynamical system with additive stochastic disturbances, which is characterized as network coherence by the Laplacian spectrum. We capture this robustness using a quantity that we call network coherence. Next, We will review the definition of first- and second-order network coherence in the system dynamics.

2.1. Coherence in networks with first-order dynamics

For the first-order consensus, at time t, each node j has a single state $x_j(t)$. And the state of the overall system at time t which is given by the vector $x(t) \in R^N$. Each node state suffers stochastic disturbances and the goal is to let the nodes maintain consensus at the mean of their current states and the dynamics of each node in the network is defined as follows:

$$\dot{x}(t) = -Lx(t) + \omega(t), \tag{1}$$

where $x(t) \in R^N$ is the state vector of network, $\omega(t)$ is an N-vector of independent Gaussian white noise stochastic processes and L is the Laplacian matrix.

Network coherence quantifies the steady-state variance of these fluctuations and it can measure the robustness of the consensus process to the additive noise. If network with small steady-state variance have high network coherence, that is it has more robust to noise than networks with low coherence [28]. One can obtain the variance of these fluctuations in the first-order consensus systems from the definition of network coherence.

The first-order network coherence is defined as the mean and steady-state variance of the deviation from the mean of the current nodes states

$$H_{FO} := \lim_{t\to\infty} \frac{1}{N} \sum_{j=1}^{N} \mathrm{var}\left\{ x_j(t) - \frac{1}{N}\sum_{k=1}^{N} x_k(t) \right\}. \tag{2}$$

The output of the system (1) can be defined as

$$y(t) = Jx(t), \tag{3}$$

where J is the projection operator $J = I - \frac{1}{N}\mathbf{1}\mathbf{1}^T$, with $\mathbf{1}$ the N-vector of all ones. H_{FO} is given by the H_2 norm of the system defined in Eqs. (1) and (3),

$$H_{FO} = \frac{1}{N}\mathrm{tr}\left(\int_0^\infty \exp(-L^T t) J \exp(-Lt) dt \right). \tag{4}$$

Suppose L is the Laplacian matrix of connected graph with eigenvalues $0 = \lambda_1 < \lambda_2 \le \cdots \le \lambda_N$. We can find that H_{FO} is fullly determined by the laplacian spectrum of L [26,27]. Thus, the first-order network coherence is

$$H_{FO} = \frac{1}{2N} \sum_{i=2}^{N} \frac{1}{\lambda_i}. \tag{5}$$

2.2. Coherence in networks with second-order dynamics

Similarly, for the second-order consensus, each node j has two state variables $x_{1,j}(t)$ and $x_{2,j}(t)$ $(j = 1, 2, \ldots, N)$. The state of the whole system is captured in $x_1(t)$ and $x_2(t)$ (N-vectors). Nodes update their states based on the states of their neighbors in the network and they are also exposure to random external disturbances that enter through the $x_2(t)$ terms. Thus, the system dynamics are

$$\begin{pmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{pmatrix} = \begin{pmatrix} 0 & I \\ -L & -L \end{pmatrix} \begin{pmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{pmatrix} + \begin{pmatrix} 0 \\ I \end{pmatrix} w(t), \tag{6}$$

where $w(t)$ is a $2N$ disturbance vector with zero-mean, uncorrelated second-order processes and unit variance. L still is the Laplacian matrix.

It is similar to the first-order network coherence and the second-order network coherence is the mean, steady-state variance of the deviation from the average of $x_1(t)$,

$$H_{SO} = \lim_{t\to\infty} \frac{1}{N} \sum_{j=1}^{N} \mathrm{var}\left\{ x_{1,j}(t) - \frac{1}{N}\sum_{k=1}^{N} x_{1,k}(t) \right\}. \tag{7}$$

The output for the system (6) can be defined as

$$y(t) = \begin{pmatrix} J & 0 \end{pmatrix} \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}, \tag{8}$$

where J is also the projection operator and the second-order network coherence is given by the H_2 norm of the system that defined by Eqs. (6) and (8). Then, the value is also fully determined by the eigenvalues of the Laplcian matrix [15]. Thus, the second-order network coherence is

$$H_{SO} = \frac{1}{2N} \sum_{i=2}^{N} \frac{1}{\lambda_i^2}. \tag{9}$$

From the above analysis, for a weighted network, the network coherence can be obtained straightforwardly from Laplacian spectrum. However, the analytical determination of this spectrum is difficult. Because the numerical calculation of Laplacian eigenvalues is limited by the order of the graph and non-practical for large network. In the next section, we calculate the sum and square sum of reciprocals of all nonzero Laplacian eigenvalues by using the recursive relationship of its eigenvalues at two successive generations and Squeeze Theorem.

3. Model description

Saad et al. proposed the hypercube is regarded as a graph. The hypercube is an undirected graph consisting of 2^g vertices labeled from 0 to $2^g - 1$ and such that there is an edge between any two