Chaos, Solitons and Fractals 115 (2018) 84–95

Contents lists available at ScienceDirect

Chaos, Solitons and Fractals

Nonlinear Science, and Nonequilibrium and Complex Phenomena

journal homepage: www.elsevier.com/locate/chaos

Chaotic properties of elementary cellular automata with majority memory

Junkang Xu[a], Erlin Li[b], Fangyue Chen[b], Weifeng Jin[c,*]

[a] College of Medical Technology, Zhejiang Chinese Medical University, Hangzhou, Zhejiang, China
[b] School of Science, Hangzhou Dianzi University, Hangzhou, Zhejiang, China
[c] College of Pharmaceutical Sciences, Zhejiang Chinese Medical University, Hangzhou, Zhejiang, China

ARTICLE INFO

Article history:
Received 16 April 2018
Revised 29 June 2018
Accepted 23 August 2018
Available online 1 September 2018

Keywords:
Majority memory
Cellular automata
Symbolic vector space
Chaos

ABSTRACT

In this paper, a practical framework of symbolic vector space is applied to uncover the time-asymptotic evolutionary behaviors of cellular automata with majority memory. This work focuses on elementary cellular automata rules with majority memory (ECAMs) and Bernoulli-shift parameters $\sigma = 1$, $\tau = 2$. The concepts of forward time$-\tau$ map and characteristic function are exploited to display the Bernoulli-shift features and modes. Particularly, it is rigorously verified that ECAMs rule 10 actually defines a Bernoulli-measure global attractor in the bi-infinite symbolic vector space. It is furthermore identified that ECAMs rule 10 possesses complicated symbolic dynamics; namely, it is endowed with temporal chaotic features as positive topological entropy and topologically mixing. Therefore, ECAMs rule 10 is chaotic on its global attractor according to definitions of both Li-York and Devaney. To this end, it should be underlined that the procedure proposed in this study is applied to other ECAMs rules with the same shifting mode, and the corresponding results are exhibited.

1. Introduction

Cellular automata (CAs), originally designated by Stanislaw Ulam and John von Neumann in the 1940s, are a group of dynamical systems whose time, space and states are all discrete. They have been confirmed useful both as theoretical models for presenting chaos and complexity of non-linear dynamics and as specific applications in a broad range of scientific fields [1]. Based on their simple structures and displaying complex emergent behaviors, CAs are widely studied by a growing number of researchers to investigate their mathematical theory and practical application [2,23,25]. For instance, CAs were studied as a particular type of topological dynamical systems in the 1960s by Gustav A. Hedlund, who established the connection between symbolic dynamics and CAs and thereby proved many results in the light of this point of view [3]. Subsequently, nonlinear dynamical properties (i.e., subshift attractors, chaotic dynamics) of some specific CAs, including surjective CAs, reversible CAs, permutive CAs, expansive CAs, and additive CAs, are deeply understood [4–9]. It is indeed that some properties are hard to check algorithmically [21,24]. Besides, some qualitative and quantitative classification schemes are proposed since the first attempt to classify the CAs rule space by Wolfram [2].

In the 2000s, the concepts and frameworks of CAs with memory (CAMs) and the corresponding distinct memory functions were originally conceived by Alonso-Sanz [10]. These memories include exponential weighing, inverse memory, parity memory, and continuous valued memory, etc. Compared with conventional and standard CAs, the new state of a cell in CAMs is determined by not only its neighborhood states at the preceding time step but also its states itself in the previous time steps. Therefore, the original local rules and the memory functions are integrated into new evolution rules for CAMs. More specifically, the memory function ϕ can be defined as $\phi(x_i^{t-\nu} \dots x_i^{t-1}\ x_i^t) \longrightarrow s_i^t$, where $\nu < t$ represents the degree of memory backwards, and the cell state s_i^t is a transition state of the $i-th$ cell position with memory backward up to a specific value ν. Subsequently, the original rule is applied as $f(\dots s_{i-1}^t\ s_i^t\ s_{i+1}^t \dots) \longrightarrow x_i^{t+1}$ to execute the subsequent evolution. Note that f in CAMs makes it act by featuring each cell by a summary of its past states from ϕ, and one can say that cells canalize memory to f [11].

Recently, many contributions about computational properties and dynamical classification of CAMs have been implemented along with other computer simulations. It is illustrated that CAMs can display complex behaviors in their time evolutions. For instance, a symbolic dynamics perspective to the empirical observations concerning the elementary cellular automata with minority memory is conducted in [15]. ECAs rules 126 and 30 with

* Corresponding author.
E-mail address: jin_weifeng@126.com (W. Jin).

https://doi.org/10.1016/j.chaos.2018.08.019

the particular majority memory functions are endowed with rich and complicated glider phenomena [12,13]. A general method of obtaining reversible elementary CAs with memory (ECAMs) from reversible and permutative elementary CAs (ECAs) is offered in [22,27]. The effect of memory embedded in cells and links on a particular two-dimensional totalistic cellular automaton is qualitatively studied in [26]. Besides, a qualitative classification of ECAMs was illustrated as strong, moderate and weak rules [11]. Thus, any ECAs class can be converted to any ECAMs class by adding an appropriate type of memory. By exploiting two fundamental homeomorphisms in symbolic vector space, all ECAMs are furthermore grouped into 88 equivalence classes in the sense that different mappings in the same equivalence class are mutually topologically conjugate [14].

The most basic problem in studying the general dynamical systems is to understand and/or to predict their long-term dynamics as $t \to \infty$. Recently, chaotic properties of Chua's Bernoulli-shift rules [18] and ECAs rule 12 with minority memory [15] are investigated under the framework of symbolic dynamics. This paper is devoted to applying these analytical methods to describe a quantitatively nonlinear dynamics perspective to ECAMs with Bernoulli-shift parameters $\sigma = 1$ and $\tau = 2$ therein. Meanwhile, this work is imposed exclusively on ECAs rule 10 with the majority memory function, and the number of cells performing memory is considered as $\nu = 3$; that is, the temporary frameworks are determined by the past three states of each cell. For this regard, the majority function and ECAs rule 10 can be represented by the Boolean truth table as $\{000 \to 0, 001 \to 0, 010 \to 0, 011 \to 1, 100 \to 0, 101 \to 1, 110 \to 1, 111 \to 1\}$, and $\{000 \to 0, 001 \to 1, 010 \to 0, 011 \to 1, 100 \to 0, 101 \to 0, 110 \to 0, 111 \to 0\}$, respectively. In light of symbolic vector space and symbolic dynamics, chaotic dynamical properties of rule 10 with majority memory $\nu = 3$ (denoted simply as ECAMs rule 10) on its global attractor are formulated in subtle detail, such as topological entropy and topologically mixing. Meanwhile, the subsystems of other ECAMs rules with the same shifting mode are exhibited along with their chaotic properties.

The rest paper is arranged as follows. In Section 2, the basic concepts of symbolic vector space and ECAMs are prepared and introduced. In Section 3, the qualitative properties of ECAMs rule 10 are exhibited by exploiting the concepts of characteristic function and forward time$-\tau$ map. The necessary and sufficient conditions are identified for a subshift of finite type of ECAMs rule 10 in the bi-infinite symbolic vector space. Moreover, it is demonstrated that this subshift is its global attractor. In Section 4, chaotic dynamics of this attractor is proved based on the existing results on subshifts of finite type of symbolic dynamics. It is indeed that ECAMs rule 10 possesses the positive topological entropy and is topologically mixing on the global attractor, and thereby it is chaotic according to definitions of both Li-York and Devaney. Meanwhile, Bernoulli-shift dynamics of ECAMs rules 2, 6, 11, 14, 34, 38, 42, 43, 46, 58, 74, 106, 130, 134, 138, 142, 162, 170 and 184 with the same shifting mode is summarized. In Section 5, the main results of this work are highlighted, and the future studies are prospected.

2. Symbolic vector space and ECAMs

For a finite symbol set S, let S^Z denote the state space consisting of all bi-infinite configurations. A definition of distance "d" on S^Z is given as

$$d(x, \bar{x}) = \sum_{i=-\infty}^{\infty} \frac{1}{2^{|i|+1}} d_i(x_i, \bar{x}_i), \tag{1}$$

where $x, \ \bar{x} \in S^Z$, and $d_i(\cdot, \ \cdot)$ is a distance function on S as $d_i(x_i, \bar{x}_i) = \begin{cases} 1, & if \quad x_i \neq \bar{x}_i \\ 0, & if \quad x_i = \bar{x}_i \end{cases}$. Now, the original symbolic space S^Z is extended to the symbolic vector space $S_n^Z = \{X = (x^{(1)^T}, x^{(2)^T}, \dots x^{(n)^T})^T | x^{(j)} \in S^Z, \ \ j = 1, 2, \dots, n\}$, where T refers to the transpose operation. Since $S_n = \prod_{i=1}^n S$ is a finite symbol set, one can use the same distance Eq. (1) on S_n^Z. It is noted that S_n^Z can be viewed as the product space of S^Z, i.e., $S_n^Z = \prod_{i=1}^n S^Z$. Therefore, S_n^Z is a compact space.

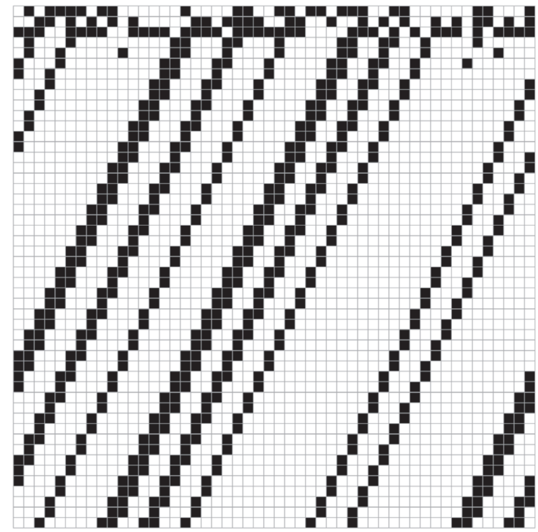

Fig. 1. Typical spatio-temporal patterns governed by ECAMs rule 10 from random initial configuration.

For any $X \in S_n^Z$, its column is denoted by $X_i = (x_i^{(1)} x_i^{(2)} \dots x_i^{(n)})^T$, and the block with size $n \times m$ over S is denoted by $\begin{pmatrix} x_i^{(1)} \dots x_{i+m-1}^{(1)} \\ \dots \\ x_i^{(n)} \dots x_{i+m-1}^{(n)} \end{pmatrix}$. Thus, the definition of left shift map σ on S_n^Z is given by $[\sigma(X)]_i = X_{i+1}$, for any $X \in S_n^Z$, $i \in Z$. In this paper, one special type of memory with $\nu = 3$ is considered. This implies that $n = 3$, and $S_n^Z = S_3^Z$. Therefore, the definition of global map F_{10} of ECAMs rule 10 is given as follows: for any $X \in S_3^Z$,

$$F_{10}: \ S_3^Z \to S_3^Z,$$

with

$$F_{10} \begin{pmatrix} x^{(1)} \\ x^{(2)} \\ x^{(3)} \end{pmatrix} = \begin{pmatrix} y^{(1)} \\ y^{(2)} \\ y^{(3)} \end{pmatrix},$$

where $y^{(1)} = x^{(2)}, \ y^{(2)} = x^{(3)}, \ y^{(3)} = f_{10} \circ \phi(\ x^{(1)^T}, \ x^{(2)^T}, \ x^{(3)^T})^T$. It follows from [14] that F_{10} and σ are continuous, and therefore (S_3^Z, F_{10}) and (S_3^Z, σ) are both compact dynamical systems. The computer simulation of ECAMs rule 10 from a random initial condition is illustrated in Fig. 1, where the white and black pixels represent symbols 0 and 1, respectively.

For a compact dynamical system (S_n^Z, F) and a subset $\Lambda \subseteq S_n^Z$, if $F(\Lambda) \subseteq \Lambda$, then Λ is F-invariant. Moreover, if Λ is closed and F-invariant, then (Λ, F) or simply Λ is called a subsystem of F. Specifically, $\bigcap_{k \geq 0} F^k(S_n^Z)$ is a subsystem and is called the global attractor of F. Besides, let $\mathscr{A}$ be a set of some finite blocks with the same size $n \times m$ over S, and

$$\Lambda = \Lambda_{\mathscr{A}} = \{X = (x^{(1)^T}, x^{(2)^T}, \dots x^{(n)^T})^T \in S_n^Z \ | \ \begin{pmatrix} x_i^{(1)} & \dots & x_{i+m-1}^{(1)} \\ \dots & \dots & \dots \\ x_i^{(n)} & \dots & x_{i+m-1}^{(n)} \end{pmatrix} \in \mathscr{A}, \ \ \forall i \in Z\}.$$

This construction method guarantees that $\Lambda_{\mathscr{A}}$ is a subsystem (or subshift) of σ, and $\mathscr{A}$ is said to be the determinative block system of Λ.