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Numerical patterns in system of integer and non-integer order derivatives

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ABSTRACT

This research work contributes to the formation of spatial patterns in fractional-order reaction-diffusion systems. The classical second-order partial derivatives in such systems are replaced with the Riemann–Liouville fractional derivative of order $\alpha \in (1, 2]$. We equally propose a novel numerical scheme for the approximation in space, and the resulting system of equations is advance in time with the improved fourth-order exponential time differencing method. Mathematical analysis of general two-component integer and non-integer order derivatives are provided. To guarantee the correct choice of the parameters in the main dynamics, we carry-out their linear stability analysis. Theorems regarding the local-stability and the conditions for a Hopf-bifurcation to occur are also provided. The proposed numerical method is applied to solve two non-integer-order models, namely the biological (predator-prey) and chemical (activator-inhibitor) systems. We observed some amazing patterns that are completely missing in the classical case at different values of fractional power α in high dimensions that evolve in fractional reaction-diffusion equations.

1. Introduction

Dynamical systems with integer and non-integer order derivatives are often known as the classical and fractional differential equations, respectively. The study of both cases have received a lot of attention over the years and across many disciplines in areas of applied sciences and engineering [1,5,8,9,19,20,21]. More importantly, they are mostly applied in the modelling of real-life dynamical systems, such as the pattern formation process, chaotic and spatiotemporal phenomena, quasi-chaotic dynamical systems, the dynamics of porous media or complex material, and random walks with memory.

Over the years, most of the patterns obtained has been examined for the class of integer-order reaction-diffusion systems in one and high dimensions. In the present work, we aim to extend this study by considering the pattern formation results of fractional-in-space reaction-diffusion systems, in a number of cases, for two different reaction kinetics, one in biology and the other with applications in physics and chemistry. We chose these dynamics to be able to draw a reasonable comparison between the integer and non-integer order systems, yet spatial patterns arising from both often appear similar [15,17].

Now, we first introduce the general two-component reaction-diffusion system, written in the form

$$\left.\begin{aligned} u_t &= d_1\Delta^2 u(x,t) + f_1(u,v), \\ v_t &= d_2\Delta^2 v(x,t) + f_2(u,v), \end{aligned}\right\} \tag{1.1}$$

where u, v are vectors representing the species concentration or densities at time t and position x in the presence of diffusions $d_1 > 0$, $d_2 > 0$. The nonlinear functions describing the reaction kinetics are given by f_1 and f_2. System (1.1) can be solved using any of the boundary conditions namely; Neumann (zero-flux), Dirichlet, periodic or Robin type on bounded domain $\Omega \subset \mathbf{R}^n$. The chemical species concentrations are specified at $t = 0$, $\forall x \in \Omega$. The choice of zero-flux boundary condition in this paper is to ensure the dynamic system is self-contained with zero population flux across the boundary.

A diffusion-driven instability which is commonly known as the Turing instability occurs when a homogeneous equilibrium state solution of system (1.1) is linearly stable to some perturbations in the absence of the diffusion terms (d_1, d_2) but linearly unstable in the presence of diffusion to small spatial perturbations.

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A spatially-uniform steady state of the system (1.1) is the state $(u^*, v^*) : f_1(u^*, v^*) = f_2(u^*, v^*) = 0$ in such a way that $u = u^*, v = v^*$ satisfies the boundary conditions. Turing demonstrated that under a particular condition on the parameter values, an equilibrium point could be linearly stable if the diffusion is absent but unstable if otherwise.

For instance, using a zero-flux boundary condition on a rectangular domain, for diffusion-driven instability to occur, the conditions

$$\left.\begin{aligned} &\tfrac{\partial f_1}{\partial u} + \tfrac{\partial f_2}{\partial v} < 0, \quad \tfrac{\partial f_1}{\partial u}\tfrac{\partial f_2}{\partial v} - \tfrac{\partial f_1}{\partial v} + \tfrac{\partial f_2}{\partial u} > 0, \quad d_1\tfrac{\partial f_2}{\partial v} + d_2\tfrac{\partial f_1}{\partial u} > 0, \\ &d_1\tfrac{\partial f_2}{\partial v} + d_2\tfrac{\partial f_1}{\partial u} > 2\sqrt{d_1 d_2\left(\tfrac{\partial f_1}{\partial u}\tfrac{\partial f_2}{\partial v} - \tfrac{\partial f_1}{\partial v} + \tfrac{\partial f_2}{\partial u}\right)} \end{aligned}\right\} \tag{1.2}$$

must be satisfied. Further analysis of two-component system can be found in [7,10,12–14,18].

Following the description for the integer-order system above, we now consider the general noninteger-order reaction-diffusion model, as a special case of (1.1) given in the form

$$\left.\begin{aligned} u_t &= \Delta^\alpha u(x,t) + \wp f_1(u,v), \\ v_t &= d\Delta^\alpha v(x,t) + \wp f_2(u,v), \end{aligned}\right\} \tag{1.3}$$

where $\wp$ denotes the relative strength of the local reaction kinetics, $1 < \alpha \le 2$ is the fractional order of the species $u(x, t)$, $v(x, t)$ densities. Other parameters remain as earlier defined. The solution of the above system can be sought, subject to any boundary conditions. For instance in one component, for an infinite model, $x \in (-\infty, \infty)$, here $\mathbf{R}$ is a subset of $(-\infty, \infty)$. For the case $x \in [0, L]$, $\frac{\partial u_i}{\partial x}(0,t) = \frac{\partial u_i}{\partial x}(L,t) = 0$, $i = 1, 2, \ldots, n$, no-flux or Neumann boundary condition for a finite system, and $x \in [0, L]$, $\mathbf{u}(0,t) = \mathbf{u}(L,t) = \mathbf{u}_a$, $i = 1, 2, \ldots, n$ is referred to as the Dirichlet boundary condition. Finally for a fixed system, where $u_i(t, \mathbf{x}) \in \mathbf{R}^n$, $\{_i : \mathbf{R}^n \to \mathbf{R}$ and d is the diffusion tensor, and $\Delta^\alpha = \left(\frac{\partial^\alpha}{\partial x^\alpha}, \frac{\partial^\alpha}{\partial y^\alpha}, \frac{\partial^\alpha}{\partial z^\alpha}\right)^{\mathbf{T}}$, is the Riemann–Louiville fractional gradient in high dimensions, for

$$\frac{\partial^\alpha}{\partial x^\alpha}u(x,y,z) = \frac{1}{\Gamma(1-\alpha)}\frac{\partial}{\partial x}\int_0^x \frac{u(\xi,y,z)}{(x-\xi)^\alpha}d\xi,$$

with $\frac{\partial^\alpha}{\partial y^\alpha}$ and $\frac{\partial^\alpha}{\partial z^\alpha}$ having the same expressions. Also, as reported in [18] we have

$$\Delta^\alpha u(x,t) = \frac{\partial^\alpha}{\partial t^\alpha}\nabla^2 u(x,t) = \mathcal{L}^{-1}\left\{\frac{\partial^{\alpha-1}}{\partial t^{\alpha-1}}\nabla^2 u(x,t)|_{t=0}\right\}$$

with similar expression for v, is the generalization of the diffusion operator from standard to fractional. When solving the system with the Laplace transform, the term $\mathcal{L}^{-1}\left\{\frac{\partial^{\alpha-1}}{\partial t^{\alpha-1}}\nabla^2 u(x,t)|_{t=0}\right\}$ actually preludes the introduction of nonphysical terms. Proceeding just as in the classical case, perturbation about point (u^*, v^*) leads to linearized system of equations

$$\left.\begin{aligned} \tfrac{\partial \Delta u(x,t)}{\partial t} &= \Delta^\alpha u(x,t) + \wp(a_{11}\Delta u + a_{12}\Delta v), \\ \tfrac{\partial \Delta v(x,t)}{\partial t} &= D\Delta^\alpha v(x,t) + \wp(a_{21}\Delta v + a_{22}\Delta u), \end{aligned}\right\} \tag{1.4}$$

By adopting the techniques of spatial Fourier and temporal Laplace transforms, we get

$$\begin{aligned} \zeta\underline{\Delta u}(\omega,\zeta) - \Delta\tilde{u}(\omega, t=0) &= \wp[a_{11}\underline{\Delta u}(\omega,\zeta) + a_{12}\underline{\Delta v}(\omega,\zeta)] \\ &\quad -\zeta^\alpha\omega^2\underline{\Delta u}(\omega,\zeta), \\ \zeta\underline{\Delta v}(\omega,\zeta) - \Delta\underline{v}(\omega, t=0) &= \wp[a_{21}\underline{\Delta u}(\omega,\zeta) + a_{22}\underline{\Delta u}(\omega,\zeta)] \\ &\quad -\zeta^\alpha\omega^2\underline{\Delta v}(\omega,\zeta), \end{aligned}$$

which decouples into

$$\underline{\Delta u}(\omega,\zeta) = \frac{(\zeta + \zeta^\alpha D\omega^2 - \wp a_{22})\overline{\Delta u}(\omega, t=0) + \wp a_{12}\overline{\Delta v}(\omega, t=0)}{(\zeta + \zeta^\alpha\omega^2 - \wp a_{11})(\zeta + \zeta^\alpha D\omega^2 - \wp a_{22}) - \wp^2 a_{12}a_{21}},$$

$$\underline{\Delta v}(\omega,\zeta) = \frac{(\zeta + \zeta^\alpha\omega^2 - \wp a_{11})\overline{\Delta v}(\omega, t=0) + \wp_{21}\overline{\Delta u}(\omega, t=0)}{(\zeta + \zeta^\alpha\omega^2 - \wp a_{11})(\zeta + \zeta^\alpha D\omega^2 - \wp a_{22}) - \wp^2 a_{12}a_{21}}.$$

Thus, the Turing condition for non-integer order reaction-diffusion system is obtained by finding the inverse of the Laplace transforms.

The aim of this paper is in folds, and is broken into sections. We introduce two important dynamics in Section 2 and present their linear stability results. Condition for the emergence of Turing instability is also provided. Numerical method for time and space discretization is formulated in Section 3. Numerical experiment at different instances of fractional order is reported in Section 4, and finally conclude the paper with the last section.

2. The dynamical systems and their stability analysis

In this section, we study the local (asymptotic) stability of the nontrivial states, existence of Hopf-bifurcation at the neighbourhood of the steady state, and examine the condition for the Turing instability to arise. Two examples that are still current interests are considered

2.1. Biological (prey-predator) system

We consider the diffusive Holling–Tanner integer and non-integer order reaction-diffusion system [14]

$$\left.\begin{aligned} \tfrac{\partial u}{\partial t} - d_1\Delta^\alpha u &= f_1(u,v) = ru\tfrac{\kappa - u}{\kappa + cu} - \tfrac{\mu uv}{u + \rho v}, \\ \tfrac{\partial v}{\partial t} - d_2\Delta^\alpha v &= f_2(u,v) = \phi v\left(1 - \tfrac{hv}{u}\right), \end{aligned}\right\} \tag{2.5}$$

where $u(x, t)$ and $v(x, t)$ describe the prey-predator population densities at time and position x. The parameters r, μ, κ, ρ, ϕ and h are nonnegative constants that denote the respective prey intrinsic growth, capturing rate, carrying capacity, half capturing saturation constant, intrinsic growth rate of the predator, rate of conversion of prey to predator population. The replacement of mass in the density at κ is given by the ration r/c, d_1 and d_2 are the diffusion coefficients of u and v, respectively. The term $\Delta^\alpha = (\partial^\alpha/\partial x^\alpha + \partial^\alpha/\partial y^\alpha)$, is the fractional Laplacian operator in two dimensions. It should be noted that we recover the classical operator when $\alpha = 2$.

Analysis of system (2.5) is done subject to the initial conditions of the form

$$u(x,y,0) > 0 \quad v(x,y,0) > 0, \quad (x,y) \in \Omega,$$

and the zero-flux boundary conditions

$$\frac{\partial u}{\partial \varpi} = \frac{\partial v}{\partial \varpi} = 0, \quad (x,y) \in \partial\Omega, \quad t > 0$$

where $\Omega \subset \mathbb{R}^\alpha$, $1 < \alpha \le 2$ is assumed to be bounded with boundary $\partial\Omega$, ϖ denotes the outward normal vector on $\partial\Omega$. Next, we verify the stability of non-diffusive form of system (2.5). That is,

$$u_t = ru\frac{\kappa - u}{\kappa + cu} - \frac{\mu uv}{u + \rho v}, \quad v_t = \phi v\left(1 - \frac{hv}{u}\right). \tag{2.6}$$

It is not difficult to see that (2.6) has a trivial state at point $E_0 = (\kappa, 0)$. Algebraic findings reveals that if $\mu < r(\rho + h)$, above model has a nontrivial state $E^* = (u^*, v^*)$, where $u^* = \frac{\kappa(\rho r + hr - \mu)}{\rho r + c\mu + hr}$, $v^* = \frac{1}{h}u^*$.