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Synchronizability and mode-locking of two scaled quadratic maps via symmetric direct-coupling

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ABSTRACT

The paper devotes to the synthesis of synchronizability and mode-locking of two scaled quadratic maps via symmetric direct-coupling. Present research shows that, similar to diffusive-coupling, the direct-coupling also admits all synchronized motions. Nevertheless, the synchronized motions are degenerated to the controlled dynamics instead of the pseudo-orbits of the local map. In consideration of chaos synchronization, nonlinear perturbations on the synchronized subspace are employed to perform the synchronization stability analysis. The synchronizability is also surveyed from a different perspective through investigating the synchronization of the coupled chaotic map in the presence of small parameter mismatch. The emergence of mode-locking phenomena in two-dimensional parameter space is secondary but proclaims the existence of incomplete synchronization.

1. Introduction

Synchronization between interacted chaotic systems has been a subject of great interest up to now. The preliminary definition of chaos synchronization is referred to as an adjusting process wherein the augmented system composed of two or more coupled chaotic systems evolves, and finally achieves an agreement of trajectories or locking of phases. The physical mechanisms of different coupling configurations have been well-understood from their theoretical studies and practical applications [1]. There are several types of interaction patterns which incorporate the uncoupled chaotic systems as an ensemble [2]. The interaction terms usually take the form of being proportional to differences of the state variables or their derivatives linearly or nonlinearly. A variety of thresholds associated with the control schemes play a crucial role in synchronizing the unrelated behavior of the augmented system [3]. The synchronized states of the augmented system share the property of asymptotic stability either locally or globally [4]. The chaos synchronization is characterized by the positive tangent Lyapunov exponents coupled with the negative transverse Lyapunov exponents, when complete synchronization is achieved [5].

From the last century to today, a large number of papers have had an in-depth insight into the synchronization of identical chaotic maps, especially one-dimensional chaotic maps, via diffusive couplings. In comparison with the uncoupled chaotic maps, such a map ensemble becomes more complex and can produce unprecedented behavior due to the dimension augment, such as on-off intermittency [6,7], spatio-temporal chaos [8], various regular spatial modes [9], riddled basins [3,7]. Much effort has been put into the study of the period-doubling (Abbr. *PD*) bifurcation scenarios, as well as the quasi-periodic behavior with mode-locking, before stepping into the aperiodic motion [10,11]. In this case, the stability conditions for the synchronized dynamics can be derived from the pseudo-orbits of the local map, especially the critical values in regard to the uniform station and the symmetry breaking/recovering [2,10]. It means that once the synchronized state has been reached, the effect of a small perturbation that destroys synchronization is rapidly damped, and synchronization will be recovered again.

An alternative pattern of interaction is direct coupling. The difference between direct coupling and diffusive coupling is that the direct coupling term is independent of the present state of the receiving oscillator [1]. This is very important in considering the oscillation death phenomenon and its applications in unidirectional synchronization of chaotic electronic circuits [1,10]. What is important is the dynamical behavior of directly coupled oscillators that may differ substantially from that of diffusively coupled ones. During the last few years, significant progress has been made in the study of the properties and the applications of interacted oscillators and map networks via direct coupling [12–18].

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The main objective of this paper is to perform the synchronization stability analysis of an ensemble map consisting of two scaled quadratic maps via symmetrical direct-coupling. The differences between diffusive coupling and direct coupling in chaotic maps are demonstrated by illustrating the unique features of the later. What is more important is that, in the direct-coupling maps, the synchronized motions are degenerated to the controlled dynamics instead of the pseudo-orbits of the local map. Despite all the fundamental difference, both coupling patterns share most commonalities. The most obvious one is that the direct-coupling also admits all synchronized motions, besides asynchronization. Furthermore, such a symmetric configuration leads to the symmetric fixed points, which may bring out in-phase and out-of-phase orbits that destine to undergo different bifurcation scenarios. One of which are interested is the generic way of the birth of Neimark-Sacker (Abbr. *NS*) bifurcation from the out-of-phase orbits. Mode-locking on the two-dimensional parameter space is secondary but demonstrates the implicit asymmetry properties behind the symmetric configuration. It is an indication of the existence of incomplete synchronization.

This paper is organized as follows: In Section 2, we introduce the model of direct-coupling maps of interest and figure out what happens on the synchronized manifold before probing into the way on achieving synchronization. The first part of Section 3 focuses on the synchronization stability of the direct-coupling maps in virtue of nonlinear perturbations. Basins of the synchronized and the non-synchronized attractors are presented to show the complexity of the synchronization in the densely chaotic regime. The second part investigates the synchronizability of the direct-coupling maps in the presence of small parameter mismatch. The acceptance region of the small symmetry-breaking bifurcation parameter are deduced from the estimations on the boundaries of the orbits. Section 4 devotes to the implicit asynchronization and demonstrates the phenomenon of mode-locking arising in the direct-coupling maps in spite of symmetry configurations. Finally, the results are summarized in the conclusions.

2. System descriptions

2.1. The scaled quadratic map

The two-dimensional map of interest is consisting of two scaled quadratic maps via symmetric direct-coupling,

$$\boldsymbol{T}(x,y): \begin{cases} x_{n+1} = g(x_n) + \varepsilon y_n \\ y_{n+1} = g(y_n) + \varepsilon x_n \end{cases} \tag{1}$$

where, $g(z) = bf_a(z)$, and $f_a(z) = 1 - az^2$. In the coupled map (1), ε is the coupling strength, b the scale factor, a the bifurcation parameter of the quadratic map $f_a(z)$. This system has an intrinsic symmetry $\boldsymbol{S}$: $(x, y) \to (y, x)$ such that $\boldsymbol{S}^{(2)}$ is the identity, which acts in the full phase space $\mathbb{R}^2$.

In the absence of the coupling strength, i.e. $\varepsilon = 0$, the factor b is undoubtedly scaling the dynamics of the quadratic map $f_a(z)$. Without considering the extreme case $b = 0$, the orbits of $g(z)$ are uniformly bounded in $[-b, b]$ if $b > 0$ or $[b, -b]$ if $b < 0$. The bifurcation values of $f_a(z)$ are denoted by a^*. All the bifurcation points of $g(z)$ are shifted from a^* to $\frac{a^*}{b^2}$. For example, the saddle-node (Abbr. *SN*) bifurcation for $g(z)$, where two fixed points are created but their stability are opposite, occurs at $-\frac{1}{4b^2}$, which is not equal to $a^* = -\frac{1}{4}$ unless $b = \pm 1$. The boundary crisis (Abbr. *BC*) curve, beyond which the dynamics approaches to infinity ultimately, is described by $ab^2 = 2$ [19,20].

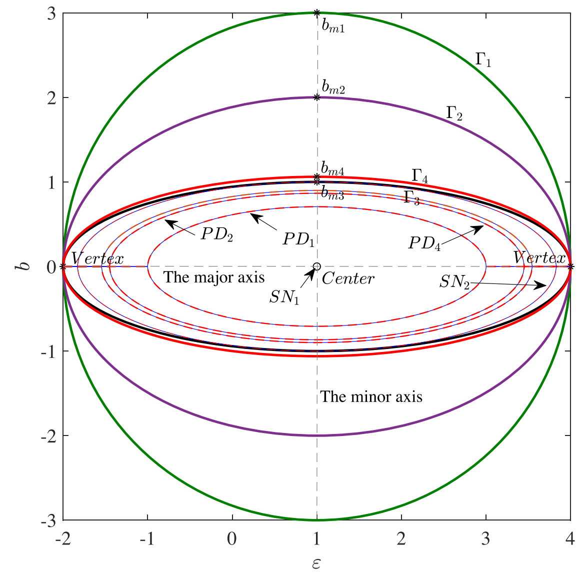

Fig. 1. Four representative *BC* curves on the $\varepsilon - b$ plane: $\Gamma_1 : a = \frac{1}{4}$, $\Gamma_2 : a = \frac{9}{16}$, $\Gamma_3 : a = \frac{9}{4}$, $\Gamma_2 : a = 2$; and several primary bifurcation curves listed in Table 1 by specifying $a = 2$.

Table 1
The elliptic equations for several primary *PD* curves, *SN* curves and *BC* curve of map $\boldsymbol{T}_1$, $a > 0$.

Type	Label	k	Elliptic equation
Fold	SN_1	0	$4ab^2 + (\varepsilon - 1)^2 = 0$
Flip	PD_1	2	$4ab^2 + (\varepsilon - 1)^2 = 4$
Flip	PD_2	$\sqrt{6}$	$4ab^2 + (\varepsilon - 1)^2 = 6$
Flip	PD_4	$\sqrt{6 + \frac{\sqrt{2 + \frac{\sqrt{2}}{165}}}{3}}$	$4ab^2 + (\varepsilon - 1)^2 = 6 + \frac{\sqrt{2 + \frac{\sqrt{2}}{165}}}{3}$
Fold	SN_2	$\sqrt{8}$	$4ab^2 + (\varepsilon - 1)^2 = 8$
BC	Γ	3	$4ab^2 + (\varepsilon - 1)^2 = 9$

2.2. Dynamics on the synchronized subspace

Given the symmetry $\boldsymbol{S}$, there is a subspace of initial conditions fixed by the symmetry. A synchronized subspace consisting of points (x, x) with $x \in \mathbb{R}$ is invariant such that the orbits starting from any points in this subspace will remain there for all time unless there are imperfections in the symmetry configurations. Clearly the diagonal line $x = y$ is that invariant manifold for complete synchronization, either chaotic orbits or periodic orbits due to the symmetry $\boldsymbol{S}$. Different from the diffusive-coupling, the dynamics of this case is governed by the following one-dimensional controlled map

$$\boldsymbol{T}_1 : z_{n+1} = bf_a(z_n) + \varepsilon z_n. \tag{2}$$

By specifying $a > 0$, the orbits of $\boldsymbol{T}_1$ are bounded in

$$\left[-\frac{2b}{\varepsilon + 2}, \frac{4ab^2 + \varepsilon^2}{4ab}\right], \text{if } b > 0, \text{or} \left[\frac{4ab^2 + \varepsilon^2}{4ab}, -\frac{2b}{\varepsilon + 2}\right], \text{if } b < 0.$$

On the parameter plane $\varepsilon - b$, all the *BC* curves, through varying the bifurcation parameter a, are fully determined by the elliptic equations in the same form,

$$4ab^2 + (\varepsilon - 1)^2 = 9. \tag{3}$$

Besides the center $(\varepsilon, b) = (1, 0)$, the vertexes on the major axis of all the *BC* curves are identical. However, the focal points of *BC* curves with different a are diverse from each other, such that the co-vertexes on the minor axis are all different. Fig. 1 displays four representative *BC* curves by specifying $a \geq \frac{1}{4}$. Therefore the major axis of the elliptic equations is $b = 0$ and the minor axis is