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A general class of multifractional processes and stock price informativeness

Qidi Peng[a,*], Ran Zhao[a,b]

[a] *Institute of Mathematical Sciences, Claremont Graduate University, United States*
[b] *Drucker School of Management, Claremont Graduate University, United States*

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ABSTRACT

We introduce a general class of stochastic processes driven by a multifractional Brownian motion (mBm) and study the estimation problems of their pointwise Hölder exponents (PHE) based on a new localized generalized quadratic variation approach (LGQV). By comparing our suggested approach with the other two existing benchmark estimation approaches (classic GQV approach and oscillation approach) through a simulation study, we show that our estimator has better performance in the case where the observed process is some unknown bivariate function of time and mBm. Such multifractional processes, whose PHEs are time-varying, can be used to model stock prices under various market conditions, that are both time-dependent and region-dependent. As an application to finance, an empirical study on modeling cross-listed stocks provides new evidence that the equity path's roughness varies via time and so does the corresponding stock price informativeness properties from global stock markets.

1. Introduction

Being a natural extension of Brownian motion (Bm) and fractional Brownian motion (fBm, see [1]), multifractional Brownian motion (mBm) has nowadays been successfully applied to many fields such as finance, network traffic, biology, geology and signal processing, etc. Unlike Bm and fBm, mBm is a continuous-time Gaussian process whose increment processes are generally not stationary. However, the feature that multifractional process allows its local Hölder regularity to change via time makes the process flexible enough to model a much larger class of empirical data than the fBm does.

In the literature, there exist several slightly different ways to define an mBm (see e.g., [2–5]). In this paper we define an mBm $\{X(t)\}_{t\in[0,1]}$ through the so-called harmonizable representation (see [2,4]): for the time index $t\in[0,1]$,

$$X(t)=\int_{\mathbb{R}}\frac{e^{it\xi}-1}{|\xi|^{H(t)+1/2}}\,\mathrm{d}\widetilde{W}(\xi), \tag{1.1}$$

where:

- H is called the pointwise Hölder exponent (PHE) of $\{X(t)\}_{t\in[0,1]}$. Recall that, for a continuous nowhere differentiable process $\{Y(t)\}_t$, its local Hölder regularity can be measured by the PHE. The PHE ρ_Y is a stochastic process defined by: for each t_0,

$$\rho_Y(t_0)=\sup\left\{\alpha\in[0,1]:\ \limsup_{\varepsilon\to 0}\frac{|Y(t_0+\epsilon)-Y(t_0)|}{|\varepsilon|^{\alpha}}=0\right\}.$$

  For the mBm $\{X(t)\}_t$, it is shown by the zero-one law (see e.g., [4]) that its PHE H is almost surely deterministic.
- The complex-valued stochastic measure $\mathrm{d}\widetilde{W}$ is defined by the Fourier transform of the real-valued Brownian measure $\mathrm{d}W$. More precisely, for all f belonging to the class of squared integrable functions over $\mathbb{R}$ (i.e., $f\in L^2(\mathbb{R})$), we have

$$\int_{\mathbb{R}}\widehat{f}(t)\,\mathrm{d}\widetilde{W}(t)=\int_{\mathbb{R}}f(t)\,\mathrm{d}W(t),$$

  where $\widehat{f}$ denotes the Fourier transform of f:

$$\widehat{f}(\xi)=\int_{\mathbb{R}}e^{-i\xi t}f(t)\,\mathrm{d}t,\ \text{ for all}\ \xi\in\mathbb{R}.$$

Multifractional processes, in particular mBm, come into vogue recently and are widely applied to financial modeling under empirical market conditions. For example, the last systemic financial crisis dated from 2007 to 2009 has strongly questioned the well-posedness of the classic dichotomy between efficient and inefficient markets. It is believed that the real financial markets are a complex system such that Bm and fBm are too reductive to explain it [6]. Unlike fBm, mBm is flexible enough to overcome this

* Corresponding author.
E-mail addresses: qidi.peng@cgu.edu (Q. Peng), ran.zhao@cgu.edu (R. Zhao).

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inconvenience, mainly because its PHE can vary via time. Through an empirical study by Bianchi and Pianese [6], it is shown that the real-world stock prices can be modeled based on an mBm. Later, by estimating the PHE of the stock price dynamics, Bianchi et al. [7] find that the PHE fluctuates around 1/2 (the sole value consistent with the absence of arbitrage), with significant deviations. In 2012, Bertrand et al. [8] introduce sparse modeling for mBm and apply it to NASDAQ time series. Recently, Bianchi and Frezza [9] have suggested a new way to quantify how far from efficiency a market is at any fixed time t. Their dynamical approach, based on estimation of the time-varying PHE of the log-variations of the 3 stock indexes - Dow Jones Industrial Average (DJIA), the Dax (GDAXI) and the Nikkei 225 (N225), allows to detect the periods in which the market itself is efficient, once a confidence interval is fixed. Note that it is more difficult to estimate the mBm's PHE than the fBm's, due to the non-stationarity of the mBm's increment processes. This problem becomes even more challenging when modeling an *individual* stock price (e.g., stock price of a particular entity) in lieu of averaged equity indexes, because the former one is not necessarily non-arbitrage and its corresponding PHE may be time-dependent and may take arbitrary values between 0 and 1. So far, there is not yet a satisfying model fitting the individual stock price process using multifractional processes. In this paper we aim to provide suitable models to describe these individual stock prices. Our main contribution consists of the following.

1. We introduce a general class of multifractional processes, that can be used to describe the behavior of individual stock returns on equity markets. The proposed model is based on the assumption that the stock return is some unknown function of both time t and an mBm at that time t (see Section 2).
2. Under the above assumption, we develop a new efficient approach to estimate the above model's PHE. The estimators from our approach are shown to be consistent (see Section 3).
3. In Section 5, through a simulation study, we compare the performances of three estimation approaches (our new localized generalized quadratic variation approach (LGQV), the classic GQV approach and the oscillation method) on various functions Φ.
4. In the empirical study (Section 6), We apply the general multifractional process to model the individual stocks and use LGQV approach to estimate cross-listed stocks' PHEs. Then we determine the market factors that drive the individual stock returns' PHEs. The estimators of the PHEs reveal that the PHEs of individual stock prices are time-varying under various market conditions and their behaviors vary via different market regions. This interesting result enables us to examine the main individual stock's PHE drivers.

Note that the Matlab codes used in Sections 5 and 6 are provided. We conclude the paper in Section 7 and provide proof of the main result in Sections 8. The supplementary graphs, tables and other detailed technical proofs are given in Appendix A.

2. A general class of multifractional processes

Before introducing the general multifractional model that we are interested in, we briefly review the estimation of the multifractional process' PHE.

In the multifractional process modeling problem, there is an obstacle: the PHE is basically not straightforwardly observed. The issue of estimating the PHE effectively arises. There are so far a number of estimation strategies existing in the literature. We refer to [10–15] and the references therein.

Coeurjolly [10,11] estimates the PHE of an mBm, starting from an observed discrete sample path of that mBm, using the LGQV approach (see also [16]). Bertrand et al. [12] study the same estimation problem as in [10,11], using the nonparametric estimation approach - increment ratio (IR) statistic method. This IR estimator has been later improved by Bardet and Surgailis [13] to the so-called pseudo-increment ratio approach, and it is applied to estimate the PHE of a more general multifractional Gaussian process (whose increments are asymptotically a multiple of an fBm) than mBm. There exist other approaches to estimate the PHE of fBm, that can be possibly extended to estimate the PHE of mBm. For example, in chaos theory and time series analysis, the statistical self-affinity is another measurement of the process path roughness. Since this exponent is tightly related to the PHE of self-similar processes (e.g., fBm), the detrended fluctuation analysis (DFA) methods developed by Peng et al. [17,18] can be used to estimate the PHE of fBm. The time-varying PHE of mBm can be then approximated by applying the DFA piecewisely over time. However, the statistical self-affinity is not equivalent to the PHE of a process, because it does not share all the properties of the Hausdorff dimension [17,18], while the Hausdorff dimension is equivalent to the PHE when the corresponding process is self-similar. In other literature, it has been shown that the wavelet-based method is actually more accurate than the DFA on estimation of the PHE. Muzy et al. [19] have obtained representations of turbulence data and Brownian signals via wavelet decompositions. Bardet et al. [20] have applied the wavelet coefficient methods to estimate the PHE of long-memory processes (e.g., fBm with its PHE being greater than 1/2), where some rate of convergence of the estimators are derived. Wendt et al. [21] have developed the wavelet leader based multifractal analysis for estimating 2D functions (images). Inspired by the above works, Jin et al. [14] have provided a wavelet-based estimator of the time-varying PHE of a class of multifractional processes with a fine convergence rate, when the observations are the wavelet coefficients of some unknown function of a multiple of mBm, i.e., the observed process is of the form $\Phi(\theta(t)X(t))$, with Φ and θ being unknown C^2-functions. In both [13,14], estimators of PHE with fine convergence rates are constructed and strategies for selecting input parameters are discussed.

Note that in our paper we also consider a model more general than the one in [14], in that it allows Φ to be a function of both t and x, i.e., we assume the observed signal is some unknown function of time t and mBm X: $\Phi(t, X(t))$. We apply the LGQV-based approach to estimate the PHE of $\Phi(t, X(t))$, when one of its discrete paths is observed. Similar to Jin et al. [14], an estimator with fine convergence rate is constructed and appropriate parameter selection is discussed. In [15,22] the oscillation estimation method, which could be applied to estimate the PHE of all processes with continuous paths, is discussed. The main advantages of our approaches are: (1) The model is simple and general enough for finance application. (2) Compared to the oscillation estimation method, the LGQV method has higher accuracy and it allows us to select the input parameter from a large range of values. We will provide a fine rate of convergence of our LGQV estimator, which will further help practitioners to determine the best input parameter values. (3) One disadvantage of the increment ratio approaches is that, it is unable to estimate the PHE over the whole time interval [0,1]. Fig. 1 is an example showing that, only part of the path of $\{H(t)\}_{t\in[0,1]}$ is estimated by the increment ratio method. However, the algorithm for LGQV-based approach can estimate H pointwisely from $t=0$ to $t=1$. Moreover, it can be easily implemented using various programming languages such as Matlab, R and Python, etc.

Throughout this paper we consider the following model: for $t\in[0,1]$,

$$Z(t) = \Phi(t, X(t)), \tag{2.1}$$

where