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Multi-sensitivity and other stronger forms of sensitivity in non-autonomous discrete systems

Mohammad Salman, Ruchi Das*

Department of Mathematics, University of Delhi, Delhi 110007, India

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ABSTRACT

In this paper we obtain different sufficient conditions for a non-autonomous discrete system to be multi-sensitivite. We study properties of a multi-sensitive non-autonomous system in detail. It is proved that on a compact metric space every finitely generated non-autonomous system which is topologically transitive having dense set of periodic points is thickly syndetically sensitive. We introduce and study the notion of totally sensitive non-autonomous systems. We also provide counter examples to support our results.

1. Introduction

Dynamical system is a very well developed branch of mathematics. In its contemporary formulation, the theory grows directly from advances in understanding complex and nonlinear systems in physics and mathematics. It describes the time dependence of a point in a geometric space and has remarkable connections with different areas of mathematics such as topology and number theory. Over the last 40 years with the discovery of chaos lots of research has been done in autonomous dynamical systems. The first paper that described chaos in a mathematically rigorous way is that of Li and Yorke [10]. Chaos theory studies the behaviour of highly sensitive dynamical systems, therefore sensitive dependence on initial conditions is an integral part in the study of non-linear science. In 1971, Ruelle introduced the concept of sensitivity for autonomous dynamical systems [14]. In 1989, Devaney introduced Devaney chaos and emphasized the significance of sensitivity in chaos [4]. Later, Li–Yorke sensitivity and n-sensitivity were introduced [1,22] and each of these described the complexity of dynamical systems. A small perturbation in the initial setup of a dynamical system leads to drastically different behaviour, therefore it is important to measure how sensitive the system is. On account of this, in 2007, Moothathu proposed stronger forms of sensitivity including multi-sensitivity [13].

Most of the investigations in dynamical systems have been done when the system is time independent. However, most of the natural phenomena involve time varying governing rules. Therefore, there is a strong need to study and develop the theory of time variant dynamical systems, that is, non-autonomous dynamical systems, which is more involved than autonomous dynamical systems. In such systems the trajectory of a point is given by successive application of different maps. It has numerous applications in biology, physics, etc., [3,5,23]. The notion of non-autonomous dynamical system was introduced by Kolyada and Snoha [9], in 1996. They extended the notion of topological entropy for autonomous dynamical system to non-autonomous dynamical system and have obtained many interesting results. Chaos and sensitivity for non-autonomous dynamical systems were introduced by Tian and Chen [19]. In 2013, authors have studied distributional chaos, sensitivity and other types of chaos in non-autonomous systems defined on a compact metric space [20]. In [18], authors have studied Spectral decomposition theorem in equicontinuous nonautonomous discrete dynamical systems. In [7,12], authors have studied stronger form of sensitivity in non-autonomous dynamical systems. Very recently, in 2018, Shao et. al., have studied Li–Yorke and distributional chaos for non-autonomous discrete systems [16].

In this paper, our main focus is on multi-sensitivity in non-autonomous dynamical systems. It is widely studied in autonomous dynamical systems and recently in 2016, many interesting results have been studied [8]. In Section 2, we give the preliminaries required for the remaining sections of this paper. In Section 3, we obtain different sufficient conditions for a non-

* Corresponding author.
E-mail addresses: salman25july@gmail.com (M. Salman), rdasmsu@gmail.com (R. Das).

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autonomous system to be multi-sensitive. In Section 4, we give equivalent conditions for multi-sensitivity in non-autonomous systems. It is proved that multi-sensitive non-autonomous system has positive topological sequence entropy. It is shown that if the family $f_{1,\infty}$ converges uniformly to f on a compact metric space, then $(X, f_{1,\infty}^{[k]})$ is multi-sensitive whenever $(X, f_{1,\infty})$ is multi-sensitive. In Section 5, we introduce the concept of total sensitivity for non-autonomous systems. It is proved that finitely generated non-autonomous system which is topologically transitive having dense set of periodic points is always thickly syndetically sensitive on a compact metric space. On a locally connected metric space equivalence of sensitivity and total sensitivity is given. We give counter examples to support our results.

2. Preliminaries

In this section, we give some definitions and results which are required for remaining sections of the paper. Throughout the paper, let $\mathbb{N}$ denote the set of natural numbers, $\mathbb{Z}_+$ denote the set of all non-negative integers and $\mathbb{Z}$ denote the set of integers.

We consider the following non-autonomous discrete dynamical system:

$$x_{n+1} = f_n(x_n), \quad n \geq 1, \tag{1}$$

where (X, d) is a metric space and $f_n: X \to X$ is a continuous map, for each $n \geq 1$. When $f_n = f$, for each $n \geq 1$, system (1) becomes an autonomous system. Denote $f_{1,\infty} := \{f_n\}_{n=1}^{\infty}$, for given positive integers i and n, $f_n^i := f_{n+i-1} \circ \cdots \circ f_n$, $f_n^0 := id$ and the kth iterate by $f_{1,\infty}^{[k]} = \{f_{k(n-1)+1}^k\}_{n=1}^{\infty}$, for any $k \in \mathbb{N}$. For the system (1), the *orbit* of any point $x \in X$ is the set, $\{f_1^n(x) : n \in \mathbb{Z}_+\} = \mathcal{O}_{f_{1,\infty}}(x)$. A point $p \in X$ is called *k-periodic* for the system $(X, f_{1,\infty})$, if $f_1^{n+k}(p) = f_1^n(p)$, for all $n \geq 0$. The set of all periodic points of the system $(X, f_{1,\infty})$ is denoted by $P(f_{1,\infty})$. Taking $f_n = f$, for all $n \geq 1$. If V is a non-empty subset of X satisfying $f_1^n(V) \subseteq V$, for each $n \geq 1$, then V is called an *invariant set* of the system $(X, f_{1,\infty})$. If V is a non-empty, closed and invariant subset of X, and no proper subset of V is non-empty, closed and invariant, then V is said to be a *minimal* subset of $(X, f_{1,\infty})$. A point $x_0 \in X$ is said to be *almost periodic point* or *minimal point* if its closure orbit is a minimal subset of $(X, f_{1,\infty})$. Equivalently for any $\epsilon > 0$, the set $\{n \in \mathbb{N} : d(f_1^n(x_0), x_0) < \epsilon\}$ has bounded gaps. Clearly, every periodic point is almost periodic. Let $(Y, g_{1,\infty})$ be another non-autonomous system, then if $\pi: X \to Y$ is continuous onto which intertwines the actions, then we say that $(Y, g_{1,\infty})$ is a factor of the system $(X, f_{1,\infty})$ and $(X, f_{1,\infty})$ is an extension of $(Y, g_{1,\infty})$. Let $B(x, \epsilon)$ denote the open ball of radius $\epsilon > 0$ and center x.

Definition 2.1 [19]**.** We say $(X, f_{1,\infty})$ *topologically transitive*, if for each pair of non-empty open subsets U, V of X, there exists $n \in \mathbb{N}$ such that $f_1^n(U) \cap V \neq \emptyset$. For any two non-empty open subsets U and V of X denote, $N_{f_{1,\infty}}(U, V) = \{n \in \mathbb{N} : f_1^n(U) \cap V \neq \emptyset\}$. Therefore, the system $(X, f_{1,\infty})$ is transitive if for any pair of non-empty open subsets U, V of X, $N_{f_{1,\infty}}(U, V) \neq \emptyset$.

Definition 2.2. A non-autonomous system $(X, f_{1,\infty})$ is *weakly mixing* if for any two pairs of non-empty open subsets U_1, U_2; V_1, V_2 in X, there exists a positive integer k, such that $f_1^k(U_i) \cap V_i \neq \emptyset$, for each $i \in \{1, 2\}$.

Definition 2.3. A non-autonomous dynamical system $(X, f_{1,\infty})$ is said to be *weakly mixing of order m* $(m \geq 2)$, if for any non-empty open subsets $U_1, U_2, \ldots, U_m, V_1, V_2, \ldots, V_m$ there is $n \in \mathbb{N}$, such that $f_1^n(U_i) \cap V_i \neq \emptyset$, for each $1 \leq i \leq m$.

A non-autonomous system $(X, f_{1,\infty})$ is said to be non-trivial weakly mixing, if there exist two non-empty open disjoint subsets of X and the system is weakly mixing. A family $f_{1,\infty}$ is said to be *commutative* if each of its member commutes with every other member of the family. In [17], Sharma and Raghav proved that if the family $f_{1,\infty}$ is commutative, then both the Definitions (2.2) and (2.3) are equivalent.

Theorem 2.1 [17]**.** *If $f_{1,\infty}$ is a commutative family, then $f_{1,\infty} \times f_{1,\infty}$ is topologically transitive if and only if $\underbrace{f_{1,\infty} \times f_{1,\infty} \times \cdots \times f_{1,\infty}}_{n-times}$ is topologically transitive, for each $n \in \mathbb{N}$.*

Definition 2.4 [19]**.** The system $(X, f_{1,\infty})$ is said to exhibit *sensitive dependence on initial conditions* if there exists $\delta > 0$ such that, for every $x \in X$ and any neighborhood U of x, there exist $y \in U$ and $n \in \mathbb{N}$ with $d(f_1^n(x), f_1^n(y)) > \delta$; $\delta > 0$ is called a constant of sensitivity. We shall denote $N_{f_{1,\infty}}(U, \delta) = \{n \in \mathbb{N} : \text{there exist } x, y \in U \text{ such that } d(f_1^n(x), f_1^n(y)) > \delta\}$, for any arbitrary open subset U of X.

Definition 2.5. A non-autonomous system $(X, f_{1,\infty})$ is said to be *chaotic in the sense of Devaney* on X if

1. It is topologically transitive on X;
2. $P(f_{1,\infty})$ is dense in X;
3. It has sensitive dependence on initial conditions on X.

Definition 2.6 [7]**.** A non-autonomous system $(X, f_{1,\infty})$ is *cofinitely sensitive* if there exists $\delta > 0$ such that for any open subset U of X, $N_{f_{1,\infty}}(U, \delta)$ is cofinite, that is, there exists $N \in \mathbb{N}$ with $N_{f_{1,\infty}}(U, \delta) \supseteq [N, \infty) \cap \mathbb{N}$.

Definition 2.7 [21]**.** A non-autonomous system $(X, f_{1,\infty})$ is *multi-sensitive*, if there exists $\delta > 0$ such that for any $m \in \mathbb{N}$ and any non-empty open subsets $V_1, V_2, \ldots, V_m$ of X, $\bigcap_{i=1}^m N_{f_{1,\infty}}(V_i, \delta) \neq \emptyset$, where $\delta > 0$ is constant of sensitivity.

Definition 2.8. A set $F \subseteq \mathbb{N}$ is called *syndetic* if there exists a positive integer a such that $\{i, i+1, \ldots, i+a\} \cap F \neq \emptyset$, for each $i \in \mathbb{N}$. A non-autonomous system $(X, f_{1,\infty})$ is *syndetically sensitive* if there exists $\delta > 0$ such that for any open subset U of X, $N_{f_{1,\infty}}(U, \delta)$ is syndetic.

Definition 2.9. A *thick set* is a set of integers that contains arbitrarily long runs of positive integers, that is, given a thick set T, for every $p \in \mathbb{N}$, there is some $n \in \mathbb{N}$ such that $\{n, n+1, n+2, \ldots, n+p\} \subseteq T$. A non-autonomous system $(X, f_{1,\infty})$ is *thickly sensitive*, if there exists $\delta > 0$ such that for any open subset U of X, $N_{f_{1,\infty}}(U, \delta)$ is thick.

Definition 2.10. A set $F \subseteq \mathbb{N}$ is *thickly syndetic*, if $\{n \in \mathbb{N} : n + j \in F, \text{ for } 0 \leq j \leq k\}$ is syndetic, for each $k \in \mathbb{N}$. A non-autonomous system $(X, f_{1,\infty})$ is *thickly syndetically sensitive*, if there exists $\delta > 0$ such that for any open subset U of X, we have $N_{f_{1,\infty}}(U, \delta)$ is thickly syndetic.

Definition 2.11. Let $S \subseteq \mathbb{N}$ and $|S|$ denote the cardinality of S. Then

$$\overline{d}(S) = \limsup_{n \to \infty} \frac{1}{n} |S \cap \{0, 1, 2, \ldots, n-1\}|$$

$$\underline{d}(S) = \liminf_{n \to \infty} \frac{1}{n} |S \cap \{0, 1, 2, \ldots, n-1\}|$$

are called the *upper density* and the *lower density* of S, respectively. If $\overline{d}(S) = \underline{d}(S) = d(S)$, then $d(S)$ is called the *density* of S.

Definition 2.12 [7]**.** A non-autonomous system $(X, f_{1,\infty})$ is said to be *topologically strongly ergodic* on X, if for any two non-empty open subsets U and V of X, the upper density of $N_{f_{1,\infty}}(U, V)$ is equal to 1.

Definition 2.13. A non-autonomous system $(X, f_{1,\infty})$ is said to be *ergodic sensitive*, if there exists $\delta > 0$ such that for any open subset U of X, $N_{f_{1,\infty}}(U, \delta)$ has positive upper density.