Chaos, Solitons and Fractals 115 (2018) 127–134

Contents lists available at ScienceDirect

Chaos, Solitons and Fractals

Nonlinear Science, and Nonequilibrium and Complex Phenomena

journal homepage: www.elsevier.com/locate/chaos

Analysis and numerical simulation of multicomponent system with Atangana–Baleanu fractional derivative

Kolade M. Owolabi [a,b]

[a] Institute for Groundwater Studies, Faculty of Natural and Agricultural Sciences University of the Free State Bloemfontein 9300, South Africa
[b] Department of Mathematical Sciences, Federal University of Technology, Akure, Ondo State PMB 704, Nigeria

ARTICLE INFO

Article history:
Received 18 June 2018
Revised 17 August 2018
Accepted 22 August 2018

MSC:
34A34
35A05
35K57
65L05
65M06
93C10

Keywords:
Mittag–Leffler
Fractional derivative
Hopf-bifurcation
Oscillations
Predator-prey
Stability analysis

ABSTRACT

In this paper, we consider the mathematical analysis and numerical simulation of time-fractional multi-component systems. Here, the classical time derivatives in such systems are replace with the Atangana–Baleanu fractional derivative in the sense of Caputo. This derivative is found useful in the sense that it combines both the non-local and nonsingular kernels in its formulation. A two-step family of Adams–Bashforth method is derived for the approximation of the Atangana–Baleanu derivative. Numerical experiments presented for different instances of α, $0 < \alpha \leq 1$ correspond to our theoretical findings.

1. Introduction

Nowadays, a new contribution has been made to the application area of fractional calculus where new differential operators with non-singular and non-local kernel are applied [2,3]. The new kernel introduced applies the so-called generalized Mittag–Leffler function as its basis, and the properties of this function make the new operators to gain some additional interesting properties that are observed in real world scenarios, for example the crossover of the mean square displacement and scaling variant. This new fractional differential operators have been applied often in various fields of science, engineering and technology since it was suggested by Atangana and Baleanu in 2016.

It has been demonstrated that modelling with the Atangana–Baleanu fractional derivative has random walk for a small time. Also, it has been observed that, the Mittag–Leffler function is an important and useful filter tool than the power and exponential law functions, which makes the Atangana–Baleanu fractional derivative in the sense of Caputo, a powerful mathematical tool to model more-complex real world problems [2,4,25].

Due to their wider applications, these operators are universally known to have given birth to fractional differential equations with no artificial singularities as in the case of the Riemann–Liouville and Caputo derivatives, due to its non-local nature. We have also seen the interest of these operators in the field of numerical analysis. Though, to approximate these derivative numerically result to various computational issue. Recently, Atangana and Owolabi [4] proposed a range of fractional Adams–Bashforth schemes for the approximation of the Caputo, Caputo–Fabrizio and Atangana–Baleanu fractional derivatives. Approximation techniques based on Fourier spectral method was suggested in [28]. Many other numerical techniques that have been used in the past are well highlighted in [21,26,27] and references therein. Thus, to accommodate readers interesting in applying these derivative to numerical analysis, we present in this paper a viable numerical scheme in connection with the Atangana–Baleanu fractional differential operator in the sense of Caputo when applied to model some real world problems arising in biology.

E-mail addresses: kmowolabi@futa.edu.ng, owolabikm@ufs.ac.za

https://doi.org/10.1016/j.chaos.2018.08.022

In what follows, we briefly highlight the definitions of the Caputo and modified Riemann–Liouville derivatives, as well as the Atangana–Baleanu derivatives with fractional order that is of interest in this paper.

Definition 1.1. According to [5,6], the fractional derivative of a continuous and $n-$time differentiable function is defined by

$$ {}^{C}\mathbf{D}_t^{\alpha}u(t) = \frac{1}{\Gamma(n-\alpha)}\int_a^t (t-y)^{n-\alpha-1}\left(\frac{d}{dy}\right)^n u(y)dy, \quad n-1<\alpha\le n. \tag{1.1}$$

Definition 1.2. The modified Riemann–Liouville fractional derivative of a function u is defined by

$$ {}^{RL}\mathbf{D}_t^{\alpha}u(t) = \frac{1}{\Gamma(n-\alpha)}\left(\frac{d}{dy}\right)^n\int_a^t (t-y)^{n-\alpha-1}[u(y)-u(a)]dy, \quad n-1<\alpha\le n. \tag{1.2}$$

Definition 1.3. A non-local and non-singular kernels of the fractional derivative suggested by Atangana and Baleanu [3] in the sense of Caputo is defined as

$$ {}^{ABC}\mathbf{D}_t^{\alpha}[u(t)] = \frac{M(\alpha)}{1-\alpha}\int_0^t u'(\xi)E_\alpha\left[-\alpha\frac{(t-\xi)^\alpha}{1-\alpha}\right]d\xi \tag{1.3}$$

where $M(\alpha)$ has the same definitions as in the case of the Caputo–Fabrizio fractional derivative [6], and E_α is a one-parameter Mittag–Leffler function given as

$$ u(z) = E_\alpha(z) = \sum_{k=0}^{\infty}\frac{z^k}{\Gamma(\alpha k+1)}, \quad \alpha>0, \quad \alpha\in\mathbb{R}, \quad z\in\mathbb{C}. \tag{1.4}$$

It should be noted that there are many other definitions of fractional derivatives that are not mentioned in the paper. In Section 2, we present the mathematical analysis of the main results based on the existence of equilibrium points and the existence of exact solution for the predator-prey system. A two-step Adams–Bashforth scheme is proposed in Section 3 for numerical approximation of the Atangana–Baleanu fractional derivative in the sense of Caputo. Numerical experiments showing the distributions of the species for different instances of fractional power α, with an extension to solve a chaotic system is given in Section 4. We conclude with Section 5.

2. Mathematical analysis of predator-prey system with local derivative

The dynamical behaviour of predator-prey system consider in this paper is given

$$\begin{aligned}
\frac{dx_1}{dt} &= a_1x_1 - \frac{g_1x_1}{\eta\nu} + \frac{g_2x_2}{(1-\eta)\nu} - a_1x_1\frac{x_1}{\eta\nu} - \frac{s_1x_1x_3}{\gamma+x_1} - z_1M_1x_1,\\
\frac{dx_2}{dt} &= a_2x_2 + \frac{g_1x_1}{\eta\nu} - \frac{g_2x_2}{(1-\eta)\nu} - a_2x_2\frac{x_2}{(1-\eta)\nu} - z_2M_2x_2,\\
\frac{dx_3}{dt} &= \frac{s_2x_1x_3}{\gamma+x_1} - \sigma x_3^2 - \rho x_3
\end{aligned}\tag{2.5}$$

subject to nonnegative initial population data $x_1(0)\ge 0, x_2(0)\ge 0$ and $x_3(0)\ge 0$ which correspond to respective prey, lower-predator and top-predator. We represent the species population densities at time t by x_1, x_2 and x_3, parameters a_1, a_2 stand for the intrinsic growth rates of x_1 in compartment 1 and 2, respectively. The environmental carrying capacity is represented by ν. The catching rate is given by $z_i, i=1,2$. The prey free zone (that is, area under protection) is denoted by η. The capturing and conversion rates of predators are given by $s_i, i=1,2$. The death rate and saturation constant are denoted by ρ and γ. Finally, M_1, M_2 denote the joint effort of lower- and top-predators to catch prey species in components 1 and 2.

A number of predator-prey models with various responses have been considered with great success [14–18], it is important to mention that the dynamical behaviour of multi-species system cannot be adequately described with the local derivatives, as revealed by many research papers, books and monographs [7–13,19,20,22–24,29,30]. As a result, the classical time derivative in system (2.5) is replaced with the fractional derivative in the form

$$\begin{aligned}
\frac{d^\alpha x_1}{dt^\alpha} &= a_1x_1 - \frac{g_1x_1}{\eta\nu} + \frac{g_2x_2}{(1-\eta)\nu} - a_1x_1\frac{x_1}{\eta\nu} - \frac{s_1x_1x_3}{\gamma+x_1} - z_1M_1x_1,\\
\frac{d^\alpha x_2}{dt^\alpha} &= a_2x_2 + \frac{g_1x_1}{\eta\nu} - \frac{g_2x_2}{(1-\eta)\nu} - a_2x_2\frac{x_2}{(1-\eta)\nu} - z_2M_2x_2,\\
\frac{d^\alpha x_3}{d^\alpha t} &= \frac{s_2x_1x_3}{\gamma+x_1} - \sigma x_3^2 - \rho x_3
\end{aligned}\tag{2.6}$$

where $\frac{d^\alpha\cdot}{dt^\alpha}$ can be approximated with any of the Caputo, Caputo–Fabrizio or the Atangana–Baleanu fractional derivatives of order $\alpha, 0<\alpha\le 1$. The aim of this work is to observe through numerical simulation if the model based on the fractional derivative will be more descriptive when compared with the one with local derivative.

This paper gives a mathematical basis for computational studies of the predator-prey system (2.6), both from biological and numerical point of views, we apply the linear stability analysis and dynamical systems theory to obtain conditions on the choice of parameters that guarantee biologically meaningful equilibria.

2.1. Existence of equilibrium points

To find the equilibrium points of system (2.6), we first assume that the given model is time-independent, such that

$$\begin{aligned}
0 &= a_1x_1 - \frac{g_1x_1}{\eta\nu} + \frac{g_2x_2}{(1-\eta)\nu} - a_1x_1\frac{x_1}{\eta\nu} - \frac{s_1x_1x_3}{\gamma+x_1} - z_1M_1x_1,\\
0 &= a_2x_2 + \frac{g_1x_1}{\eta\nu} - \frac{g_2x_2}{(1-\eta)\nu} - a_2x_2\frac{x_2}{(1-\eta)\nu} - z_2M_2x_2,\\
0 &= \frac{s_2x_1x_3}{\gamma+x_1} - \sigma x_3^2 - \rho x_3
\end{aligned}\tag{2.7}$$

We obtain the point $S_0(0,\ 0,\ 0)$ which corresponds to the total wash-out of the species. For the second equilibrium point $S_1(\hat{x}_1,0,0)$, where $\hat{x}_1 = \eta\nu - \frac{\eta\nu}{a_1}\left(z_1M_1 + \frac{g_1}{\eta\nu}\right)$. The condition $a_1 - z_1M_1 > \frac{g_1}{\eta\nu}$ must be satisfied for $\hat{x}_1$ to be positive. The third point $S_2(\bar{x}_1,0,\bar{x}_3)$ indicates the presence of the top predator x_3, prey x_1 and absence of lower-predator x_2, where $\bar{x}_3 = \left(\frac{s_2\bar{x}_1}{\gamma\bar{x}_1} - \rho\right)1/\sigma$. Also for $\bar{x}_3$ to be positive, we assume that $\frac{s_2\bar{x}_1}{\gamma\bar{x}_1} > \rho$. From the first equation in (2.6), we let

$$H_1(\bar{x}_1)^2 + H_2\bar{x}_1 + H_3 = 0$$

with $H_1 = a_1 > 0$, $H_2 = a_1\gamma - \eta\left(a_1 - z_1M_1 - \frac{g_1}{\eta\nu}\right)$ and $H_3 = \left(a_1 - z_1M_1 - \frac{g_1}{\eta\nu}\right)\eta\nu\gamma - s_1\bar{x}_3\eta\nu$. Therefore, $\eta\left(a_1 - z_1M_1 - \frac{g_1}{\eta\nu}\right) > a_1\gamma$ and $s_1\bar{x}_3\eta\nu > \eta\nu\gamma\left(a_1 - z_1M_1 - \frac{g_1}{\eta\nu}\right)$ for $\bar{x}_1$ to be unique and positive.

Lastly, we consider the nontrivial case $S_3(x_1^*, x_2^*, x_3^*)$, where

$$x_2^* = \frac{\nu(1-\eta)}{(a_2+g_2)}\left\{(a_2 - z_2M_2) + \frac{g_1x_1^*}{\eta\nu}\right\},\ x_3^* = \left(\frac{s_2x_1^*}{\gamma x_1^*} - \rho\right)1/\sigma.$$

For x_2^*, x_3^* to be positive, condition $\frac{s_2x_1^*}{\gamma x_1^*}$ must hold, and by using the first equation in system (2.6), x_1^* is unique and positive if $a_1 = \frac{\eta\nu\mathfrak{G}}{\gamma}$ and $x_3^* = \frac{\gamma\mathfrak{G}}{s_1}$, where $\mathfrak{G} = a_1 - z_1M_1 - \frac{g_1}{\eta\nu} - \frac{g_2x_2^*}{(1-\eta\nu)}$. The interior case is only meaningful in the context of biology to demonstrate the coexistence and persistence of the species, hence we