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Numerical patterns in reaction–diffusion system with the Caputo and Atangana–Baleanu fractional derivatives

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ABSTRACT

In this paper, we numerically study a nonlinear time-fractional reaction-diffusion equation involving the Caputo and Atangana–Baleanu fractional derivatives of order $\alpha \in (0, 1)$. A novel algorithm known as the Laplace Adams–Bashforth method is formulated for the approximation of these derivatives. In the simulation framework, a tri-tropic food chain system is considered in which the classical time-derivatives are replaced with non-integer order derivatives. Mathematical analysis of the main system is examined for both stability and Hopf-bifurcations to occur. Numerical simulation results show the existence of chaotic behaviours and spatiotemporal oscillations as well as the emergence of some Turing patterns (such as, spots and stripes) in two-dimensional space.

1. Introduction

Nonlinear phenomena arise in a variety of apparently different contexts in real-life, for instance in applied biology, physics, engineering and sciences. Majority of nonlinear problems exist in the form of partial differential equations (PDEs). We refer to a popular class of PDEs as those that modelled nonlinear reaction–diffusion scenarios. Nonlinear diffusion problem is regarded as an important class of parabolic equations that is encountered in many physical (real-life) situations, such as the dynamics of biological systems, image processing, groundwater processes, fractals, computer vision, phase transition in electrical–electronic, and mechanical engineering [2,3,11–13,31,34–36].

Fractional reaction–diffusion systems are often applied to represent a lot of applications like the study of chemical reaction, propagation phenomena, transport system, pattern formation processes, chaos in finance and spatiotemporal distribution of species, among many others, see [19–30,32] and references therein [20].

The nonlinear time-fractional reaction–diffusion equation considered in this work is presented in a general form

$$\frac{\partial^\alpha u(x,t)}{\partial t^\alpha} = \mathbf{L}u(x,t) + \mathbf{N}u(x,t), \tag{1.1}$$

where $\mathbf{L}$ and $\mathbf{N}$ are the respective linear and nonlinear operators [5,17,18]. The fractional time derivative of order α is replace with either the Caputo or the Atangana–Baleanu fractional derivative of noninteger order. Different numerical approaches have been proposed for the solution of Eq. (1.1), among which are the homotopy perturbation transform method [7], Laplace homotopy analysis method [14], first integral method [37,38], and Fourier spectral methods [22,32] among many others.

In what follows, we quick give a tour of some important properties of fractional differentiation. The left and right Caputo fractional derivatives of order $\alpha > 0$ for a given function $f(t)$, $t \in (a, b)$ is respectively defined as

$${}^{C}\mathcal{D}^{\alpha}_{a,t} f(t) = \mathcal{D}^{\alpha-n}_{a,t}\left[u^{(n)}(t)\right]$$

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$$= \frac{1}{\Gamma(n-\alpha)}\int_a^t (t-\xi)^{n-\alpha-1}u^{(n)}(\xi)d\xi, \tag{1.2}$$

and

$$^{C}\mathcal{D}_{t,b}^{\alpha}f(t) = \frac{(-1)^n}{\Gamma(n-\alpha)}\int_t^b (\xi-t)^{n-\alpha-1}f^{(n)}(\xi)d\xi, \tag{1.3}$$

for $n > 0$ which holds for $n-1 < \alpha \le n$.

The Liouville–Caputo fractional derivative with order $\alpha > 0$ is defined as [27,33]

$$^{C}_{0}\mathcal{D}_t^{\alpha}f(t) = \frac{1}{\Gamma(n-\alpha)}\int_0^t (t-\xi)^{n-\alpha-1}f^{(n)}(\xi)d\xi. \tag{1.4}$$

The Laplace transform of (1.4) is given as

$$\mathscr{L}\left\{^{C}_{0}\mathcal{D}_t^{(\alpha)}f(t)\right\} = p^{\alpha}F(s) - \sum_{k=0}^{n-1}p^{\alpha-k-1}f^{(k)}(0), \tag{1.5}$$

where $n = [\mathbb{R}(\alpha)] + 1$.

Let $f \in H^1(a,\ b), a < b, 0 \le \alpha \le 1$, the Atangana–Baleanu fractional derivative in Caputo sense, or the left Caputo fractional derivative with Mittag–Leffler (nonlocal and nonsingular) kernel is defined by [1]

$$^{ABC}\mathcal{D}_{a,t}^{\alpha}f(t) = \frac{M(\alpha)}{1-\alpha}\int_a^t E_{\alpha}\left[-\frac{\alpha(t-\xi)^{\alpha}}{1-\alpha}\right]f'(\xi)d\xi \tag{1.6}$$

where $M(\alpha)$ is a normalized positive function that satisfies $M(0) = 1, M(1) = 1$ and $E_{\alpha}[\xi]$ denotes a one parameter Mittag–Leffler function, given in series expansion form

$$E_{\alpha}(z) = \sum_{k=0}^{\infty}\frac{z^k}{\Gamma(\alpha k+1)}, \quad \alpha > 0. \tag{1.7}$$

The Laplace transform of (1.6) is defined as [1,8–10,27]

$$\begin{aligned}\mathscr{L}\left\{^{ABC}_{a}\mathcal{D}_t^{(\alpha)}u(t)\right\}(p) &= \frac{M(\alpha)}{1-\alpha}\mathscr{L}\left\{\int_a^t u'(\xi)E_{\alpha}\left[-\alpha\frac{(t-\xi)^{\alpha}}{1-\alpha}\right]d\xi\right\}\\ &= \frac{M(\alpha)}{1-\alpha}\,\frac{p^{\alpha}\mathscr{L}\{u(t)\}(p) - s^{\alpha-1}u(0)}{p^{\alpha}+\frac{\alpha}{1-\alpha}}.\end{aligned} \tag{1.8}$$

Let $f \in H^1(a,\ b), a < b, 0 < \alpha < 1$, the Atangana–Baleanu fractional derivative in Riemann–Liouville sense or the left Riemann–Liouville derivative with Mittag–Leffler (nonlocal and nonsingular) kernel is defined by [1]

$$^{ABR}\mathcal{D}_{a,t}^{\alpha}f(t) = \frac{M(\alpha)}{1-\alpha}\frac{d}{dt}\int_a^t E_{\alpha}\left[-\frac{\alpha(t-\xi)^{\alpha}}{1-\alpha}\right]f(\xi)d\xi. \tag{1.9}$$

The aim of this paper is broken down into the following sections. Numerical schemes for the approximation of Atangana–Baleanu fractional derivative in the sense of Caputo is derived in Section 2. The main equation involving a tri-tropic food chain model is introduced and analyzed for stability in Section 3. Numerical experiment results in one and two dimensions are given in Section 4 to depict the behaviour of such dynamical system. Conclusion is finally drawn.

2. Numerical scheme for fractional-order reaction–diffusion equation

Let us consider the general partial differential equation given in (1.1). By taking the Laplace transform of both sides, we get

$$\mathcal{L}\left\{\frac{\partial^{\alpha}u(x,t)}{\partial t^{\alpha}}\right\} = \mathcal{L}\{\mathbf{L}u(x,t) + \mathbf{N}u(x,t)\}, \tag{2.10}$$

Starting with the Caputo derivative, the above expression transforms to

$$^{C}_{a}\mathcal{D}_t^{\alpha}u(s,t) = \mathcal{G}(u,t), \tag{2.11}$$

where $u(t) = u(s,t)$ and $\mathcal{G}(u,t) = \mathcal{L}\{\mathbf{L}u(x,t) + \mathbf{N}u(x,t)\}$. By following [4,6], we apply the Caputo fractional integral on both sides of (2.11) to get

$$u(t) - u(t_0) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-\xi)^{\alpha-1}\mathcal{G}(u,\xi)d\xi. \tag{2.12}$$

With $t = t_n$ in (2.12), we obtain

$$u(t_n) = u_0 + \frac{1}{\Gamma(\alpha)}\int_0^{t_n}(t_n-\xi)^{\alpha-1}\mathcal{G}(u,\xi)d\xi \tag{2.13}$$

similarly, when $t = t_{n+1}$ one obtains

$$u(t_{n+1}) = u_0 + \frac{1}{\Gamma(\alpha)}\int_0^{t_{n+1}}(t_{n+1}-\xi)^{\alpha-1}\mathcal{G}(u,\xi)d\xi \tag{2.14}$$

On subtraction, we have

$$\begin{aligned}u_{n+1} - u_n = 7\frac{1}{\Gamma(\alpha)}\bigg\{&\int_0^{t_{n+1}}(t_{n+1}-\xi)^{\alpha-1}\mathcal{G}(u,\xi)d\xi\\ &-\int_0^{t_n}(t_n-\xi)^{\alpha-1}\mathcal{G}(u,\xi)d\xi\bigg\}.\end{aligned} \tag{2.15}$$

Bear in mind that

$$\int_0^{t_{n+1}}(t_{n+1}-\xi)^{\alpha-1}\mathcal{G}(u,\xi)d\xi = \sum_{k=0}^{n}\int_{t_k}^{t_{k+1}}(t_{k+1-\xi})^{\alpha-1}\mathcal{G}(u,\xi)d\xi. \tag{2.16}$$

Next, we apply the Lagrange polynomial to approximate $\mathcal{G}(u,t)$ as

$$P(t)[\approx \mathcal{G}(u,t)] = \frac{t-t_{n-1}}{t_n-t_{n-1}}\mathcal{G}(u,t_n) + \frac{t-t_n}{t_{n-1}-t_n}\mathcal{G}(u,t_{n-1})$$

which in more compact form becomes

$$P(t) = \frac{t-t_{n-1}}{t_n-t_{n-1}}\mathcal{G}_n + \frac{t-t_n}{t_{n-1}-t_n}\mathcal{G}_{n-1}$$

It is allowed to write the first integral in (2.15) as

$$\begin{aligned}&\int_0^{t_{n+1}}(t_{n+1}-\xi)^{\alpha-1}\mathcal{G}(u,\xi)d\xi\\ &= \sum_{k=0}^{n}\int_{t_k}^{t_{k+1}}(t_{k+1}-t)^{\alpha-1}\left[\frac{t-t_{n-1}}{t_n-t_{n-1}}\mathcal{G}_n + \frac{t-t_n}{t_{n-1}-t_n}\mathcal{G}_{n-1}\right]dt\\ &= \sum_{k=0}^{n}\left[\frac{\mathcal{G}_n}{\hbar}\int_{t_k}^{t_{n+1}}(t_{n+1}-t)^{\alpha-1}(t-t_{n-1})dt\right.\\ &\left.\quad -\frac{\mathcal{G}_{n-1}}{\hbar}\int_{t_k}^{t_{n+1}}(t_{n+1}-t)^{\alpha-1}(t-t_n)dt\right]\end{aligned} \tag{2.17}$$

With substitutions $\tau = t_{n+1} - t$, $t = t_{n+1}\tau$ and $dt = -d\tau$, we have

$$\begin{aligned}&\int_{t_k}^{t_{k+1}}(t_{n+1}-t)^{\alpha-1}(t-t_{n-1})dt\\ &= \int_{tn+1-t_k}^{t_{n+1}-tk+1}\tau^{\alpha-1}(-\tau+t_{n+1}-t_{n-1})d\tau\\ &= \frac{1}{\alpha+1}\left[(t_{n+1}-t_{k+1})^{\alpha+1} - (t_{n+1}-t_k)^{\alpha+1}\right]\\ &\quad -\frac{2\hbar}{\alpha}\left[(t_{n+1}-t_{k+1})^{\alpha} - (t_{n+1}-t_k)^{\alpha}\right]\end{aligned} \tag{2.18}$$

In a similar fashion, we have

$$\begin{aligned}&\int_{t_k}^{t_{k+1}}(t_{n+1}-t)^{\alpha-1}(t-t_n)dt\\ &= -\int_{tn+1-t_k}^{t_{n+1}-tk+1}\tau^{\alpha-1}(-\tau+t_{n+1}-t_n)d\tau\end{aligned}$$