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On the solutions of fractional-time wave equation with memory effect involving operators with regular kernel

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ABSTRACT

In this paper, we give analytical solutions of a fractional-time wave equation with memory effect and frictional memory kernel of Mittag–Leffler type via the Atangana–Baleanu fractional order derivative. The method of separation of variables and the Laplace transform has been used to obtain the exact solutions for the fractional order wave equations. Additionally, we present analytical solutions considering the Caputo–Fabrizio fractional derivative with exponential kernel. We showed that the solutions obtained via Caputo–Fabrizio fractional order derivative were a particular case of the solutions obtained with the new fractional derivative based in the Mittag–Leffler law.

1. Introduction

In recent years there has been a high level of interest in the field of fractional differential equations due to their important applications in viscoelastic materials, economy, diffusion, electrical circuits, biology, chaos theory and engineering [1–19]. In particular, the fractional diffusion-wave equation is significant to applications of fractional differential equations. In [20,21], the authors generalized the classical diffusion and wave equations; different physical process such that (slow diffusion, classical diffusion, diffusion-wave hybrid and the classical wave equation) can be obtained changed the fractional order in the range $\gamma \in (0;\ 2]$. Chen in [22]analyzed and derived the solution of the time-fractional telegraph equation with three kinds of nonhomogeneous boundary conditions, using the method of separating variables. Momani in [23] suggested analytic and approximate solutions of the space- and time-fractional telegraph differential equations by means of the so called Adomain decomposition method. Huang in [24] derived the analytical solution for three basic problems of the so-called time-fractional telegraph equations. Tomovski in [25] solved the fractional wave equation with frictional memory kernel of Mittag–Leffler type via Liouville–Caputo fractional derivative. The method of separation of variables and Laplace transform was used to solved the equations. Delic in [26] studied the time fractional wave equation with Dirac delta distribution and with homogeneous initial-boundary conditions. The rate of convergence in special discrete energetic Sobolev norms are obtained. The authors in [27], proposed a numerical scheme for solving the time fractional telegraph via Liouville–Caputo derivative, a combination of method of line and group preserving scheme was proposed to find the approximate solutions. The authors in [28] based on the standard Galerkin finite element method, a fully discrete finite element scheme was presented for solving the variable coefficient fractional diffusion-wave equation, the error estimates were established and numerical experiments were included to support the theoretical results. In [29], the authors studied the telegrapher's equation considering the topological generalization of the elementary circuit used in transmission line modeling in order to include the effects of charge accumulation along the line. Capacitive and inductive phenomena were assumed to display hereditary effects modeled by the use of FC. Tomovski in [30] considered the wave equation for a vibrating string in the presence of a fractional friction with power-law memory kernel. Exact solutions were obtained in terms of the Mittag–Leffler type functions.

These models have been extended to the scope of fractional derivatives using Riemann–Liouville, Riesz and Liouville–Caputo derivatives with fractional order [31–36], these operators are based in the power law. Caputo and Fabrizio have suggested an alternative concept of differentiation using the exponential decay as kernel instead of the power law [37–41]. Recently, a new concept of differentiation was suggested with non-local and non-singular

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kernel [42]. Many studies have been done around this new suggested derivative with great results [43–49].

In this paper, we considered the fractional derivatives with Mittag–Leffler and exponential kernel in the Liouville–Caputo sense to obtain analytical solutions of the fractional wave equation with a frictional kernel of Mittag–Leffler type. This frictional memory kernel can represent many different forms of friction, which can be used depending on the properties of the environment. Based on the method of separation of variables and the Laplace transform we obtain these novel exact solutions for the fractional order wave equations.

This paper is organized as follows. In Section 2 we recall the fractional operator of type Caputo–Fabrizio and Atangana–Baleanu in Liouville–Caputo sense. The analytical solutions obtained for the fractional wave equations are given in Section 3. Conclusions are given in Section 4.

2. Fractional operators

The Liouville–Caputo fractional order derivative (C) for $(\gamma > 0)$ is defined as follows [50]

$$ {}_0^C\mathcal{D}_t^\gamma f(t) = \frac{1}{\Gamma(n-\gamma)}\int_0^t \frac{f^{(n)}(\eta)}{(t-\eta)^{\gamma-n+1}}d\eta, \quad n-1<\gamma<n, \quad (2.1)$$

where, $f^{(n)}$ is the derivative of integer nth order of $f(t)$, $n = 1, 2, \dots, \in N$.

The Caputo–Fabrizio fractional order derivative in Liouville–Caputo sense (CFC) for $(\gamma > 0)$ is given by [40]

$$ {}_0^{CFC}\mathcal{D}_t^\gamma f(t) = \frac{M(\gamma)}{n-\gamma}\int_0^t f^{(n)}(\eta)e^{-\frac{\gamma}{1-\gamma}(t-\eta)}d\eta, \qquad n-1<\gamma<n, \quad (2.2)$$

where, $M(\gamma)$ is a constant of normalization that depend of γ, which satisfies that, $M(0) = M(1) = 1$.

If, $n \geq 1$ and $\gamma \in [0, 1]$ the ${}_0^{CFC}\mathcal{D}_t^{(\gamma+n)} f(t)$ of fractional order $(n+\gamma)$ is defined as

$$ {}_0^{CFC}\mathcal{D}_t^{\gamma+n} f(t) :={}_0^{CFC}\mathcal{D}_t^{(\gamma)}({}_0^{CFC}\mathcal{D}_t^{(n)} f(t)). \quad (2.3)$$

Theorem 2.1. *If f is a function sufficiently well behaved, which supports $(n+1)$ derivatives, and $\gamma \in (0, 1)$ we have*

$$ \mathscr{L}\left\{ {}_0^{CFC}\mathcal{D}_t^{\gamma+n} f(t)\right\}(s) = \frac{s^{n+1}\mathscr{L}\{f(t)\}(s) - s^n f(0) - \dots - f^{(n)}(0)}{s+\gamma(1-s)}. \quad (2.4)$$

Proof. Considering the Eq. (2.3) and the convolution theorem we have

$$ \begin{aligned} \mathscr{L}\{ {}_0^{CFC}\mathcal{D}_t^{n+\gamma} f(t)\}(s) &= \mathscr{L}\left\{\frac{1}{1-\gamma}\int_0^t f^{(n+1)}e^{-\frac{\gamma}{1-\gamma}(t-\eta)}\right\}(s) \\ &= \frac{1}{1-\gamma}\mathscr{L}\{f^{(n+1)}\}(s)\cdot\mathscr{L}\left\{e^{-\frac{\gamma}{1-\gamma}t}\right\}(s) \\ &= \frac{1}{1-\gamma}\frac{1}{s+\frac{\gamma}{1-\gamma}}\big(s^{n+1}\mathscr{L}\{f(t)\}(s) \\ &\quad - s^n f(0) - \dots - f^{(n)}(0)\big) \\ &= \frac{s^{n+1}\mathscr{L}\{f(t)\}(s) - s^n f(0) - \dots - f^{(n)}(0)}{s+\gamma(1-s)}. \end{aligned} \quad (2.5)$$

This complete the proof. □

The fractional integral of type CF is given by [37]

$$ {}_0^{CF}I^\gamma f(t) = (1-\gamma)u(t) + \gamma\int_0^t u(\eta)d\eta. \quad (2.6)$$

Theorem 2.2. *Let $0 < \gamma \leq 1$, the following relation is obtained*

$$ {}_0^{CF}I^\gamma \; {}_0^{CFC}D^\gamma f(t) = u(t) - u(0), \quad (2.7)$$

where, ${}_0^{CFC}D^\gamma f(t) := u(t)$.

Proof. If ${}_0^{CFC}D^\gamma f(t) := u(t)$, thus

$$ \begin{aligned} {}_0^{CF}I^\gamma \; {}_0^{CFC}D^\gamma f(t) &= {}_0^{CF}I^\gamma u(t) \\ &= (1-\gamma)\varphi(t) + \gamma\int_0^t \varphi(s)ds, \end{aligned} \quad (2.8)$$

where, ${}_0^{CFC}D^\gamma u(t) = \varphi(t)$, therefore, the Eq. (2.8) it transforms in

$$ \begin{aligned} {}_0^{CF}I^\gamma \; {}_0^{CFC}D^\gamma f(t) &= (1-\gamma)\; {}_0^{CFC}D^\gamma u(t) \\ &\quad + \frac{\gamma}{1-\gamma}\int_0^t\int_0^s u'(\xi)e^{-\frac{\gamma}{1-\gamma}(s-\xi)}d\xi ds, \end{aligned} \quad (2.9)$$

considering the change in the order of integration and using the Fubini theorem, the above equation becomes

$$ \begin{aligned} &{}_0^{CF}I^\gamma \; {}_0^{CFC}D^\gamma f(t) \\ &= (1-\gamma)\; {}_0^{CFC}D^\gamma u(t) + \frac{\gamma}{1-\gamma}\int_0^t u'(\xi)e^{\frac{\gamma}{1-\gamma}\xi}\int_\xi^t e^{-\frac{\gamma}{1-\gamma}s}dsd\xi, \\ &= (1-\gamma)\; {}_0^{CFC}D^\gamma u(t) + \frac{\gamma}{1-\gamma}\int_0^t u'(\xi)e^{\frac{\gamma}{1-\gamma}\xi} \\ &\times\left[-\frac{1-\gamma}{\gamma}\left(e^{-\frac{\gamma}{1-\gamma}t} - e^{-\frac{\gamma}{1-\gamma}\xi}\right)\right]d\xi, \\ &= (1-\gamma)\; {}_0^{CFC}D^\gamma u(t) - \int_0^t u'(\xi)e^{-\frac{\gamma}{1-\gamma}(t-\xi)}d\xi + \int_0^t u'(\xi)d\xi, \\ &= u(t) - u(0). \end{aligned}$$

This complete the proof. □

The Atangana–Baleanu fractional derivative in the Liouville–Caputo sense (ABC) is defined as follows [42]

$$ {}_0^{ABC}\mathcal{D}_t^\alpha\{f(t)\} = \frac{B(\alpha)}{n-\alpha}\int_0^t \frac{d^n}{dt^n}f(\theta)E_\alpha\left[-\alpha\frac{(t-\theta)^\alpha}{n-\alpha}\right]d\theta, \quad (2.10)$$

where, $B(\alpha)$ is a normalization function, $B(0) = B(1) = 1$.

If, $0 < \alpha \leq 1$, then we define the Laplace transform for the Atangana–Baleanu fractional derivative as follows [42]

$$ \mathscr{L}\left[{}_0^{ABC}D_t^\alpha f(t)\right](s) = \left(\frac{s^\alpha\mathscr{L}[f(t)](s) - s^{\alpha-1}[f(0)]}{s^\alpha(1-\alpha)+\alpha}\right). \quad (2.11)$$

The Mittag–Leffler function of one parameter $E_\gamma(z)$ is given by [50]

$$ E_\gamma(z) = \sum_{k=0}^\infty \frac{z^k}{\Gamma(\gamma k+1)}, \quad z\in\mathbb{C}, \quad \Re(\gamma) > 0, \quad (2.1)$$

where, $\Gamma(\cdot)$ denotes the Gamma function.

The Mittag–Leffler function of two parameters is defined by the following series when the real part of γ is strictly positive [50]

$$ E_{\gamma,\beta}(z) = \sum_{k=0}^\infty \frac{z^k}{\Gamma(\gamma k+\beta)}, $$

$$ z, \ \gamma, \quad \beta\in\mathbb{C}, \quad \Re(\gamma) > 0, \quad \textit{and} \quad \Re(\beta) > 0. \quad (2.2)$$

The Srivastava–Tomovski operator is defined in the following way [25]

$$ \left(\varepsilon_{a+;\alpha,\beta}^{\omega;\gamma,\delta}\,\varphi\right)(t) = \int_a^t (t-\eta)^{\beta-1}E_{\alpha,\beta}^{\gamma,\delta}(\omega(t-\eta)^\alpha)\varphi(\eta)d\eta, \quad (2.3)$$

if, $a = 0$, the Eq. (2.3) is a convolution of functions of the form

$$ \left(\varepsilon_{0;\alpha,\beta}^{\omega;\gamma,\delta}\,\varphi\right)(t) = \int_0^t (t-\eta)^{\beta-1}E_{\alpha,\beta}^{\gamma,\delta}(\omega(t-\eta)^\alpha)\varphi(\eta)d\eta, \quad (2.4)$$

if, $\omega = 0$ and $a = 0$, the Eq. (2.3) coincides with the Riemann–Liouville fractional integral operator of order β, that is

$$ \left(\varepsilon_{0;\alpha,\beta}^{0;\gamma,\delta}\,\varphi\right)(t) = {}_0^{RL}I^\beta\varphi(t). \quad (2.5)$$