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Chaotic behaviour in system of noninteger-order ordinary differential equations

Kolade M. Owolabi [a,b,*], Abdon Atangana [a]

[a] Institute for Groundwater Studies, Faculty of Natural and Agricultural Sciences, University of the Free State, Bloemfontein 9300, South Africa
[b] Department of Mathematical Sciences, Federal University of Technology, Akure, Ondo State PMB 704, Nigeria

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ABSTRACT

The intra-specific relation two predators and a prey dependent food chain system is considered in this paper. To explore the dynamic richness of such system, we replace the classical time-derivative with either the Caputo or the Atangana–Baleanu fractional derivative operators. Two notable numerical schemes for the approximation of such derivatives are formulated. Local and global stability analysis are investigated to ensure the correct choice of the biologically meaning parameters. The condition for occurrence of the Hopf-bifurcation is also observed. We justify the performance of these schemes by reporting their absolute error when applied to nonlinear fractional differential equations. In addition, numerical simulations with different α values and experimented parameter values confirm the analytical results shows that modelling with fractional derivative could give rise to a more richer chaotic dynamics.

1. Introduction

The study of fractional calculus is not new, it has the same history as that of standard calculus in reality. In the last few decades, fractional calculus is gaining weight and undergoing a tremendous development in modelling and application to real-life cases, see [1–3,6,8–10,14,20–29,33–36], and references therein.

The nonlocality nature of the fractional derivative makes fractional problems to be more practical and accurate when compared to the classical cases, especially for models which involve memory. In recent years, efforts have been made in no small measure to formulate new fractional derivative with nonlocal and nonsingular kernels, See the work of Atangana and Baleanu [1] and that of Caputo and Fabrizio [5] for details.

Fractional calculus as its name reads, refers to fractional differentiation and integration. Fractional integration is usually refers to as the Riemann–Liouville integral, whereas there are various kinds of fractional derivatives, among which are some of the well-known definitions that was introduced in [1–3,13,31,32].

The left and right Grünwald–Letnikov derivatives with fractional order $\alpha > 0$ of function $u(t)$, $t \in (a, b)$ are respectively defined as

$$ {}^{GL}\mathcal{D}^{\alpha}_{a,t}u(t) = \lim_{h\to 0, Nh=t-a} h^{-\alpha}\sum_{j=0}^{N}(-1)^j\binom{\alpha}{j}u(t-jh), \tag{1.1}$$

and

$$ {}^{GL}\mathcal{D}^{\alpha}_{t,b}u(t) = \lim_{h\to 0, Nh=b-t} h^{-\alpha}\sum_{j=0}^{N}(-1)^j\binom{\alpha}{j}u(t+jh). \tag{1.2}$$

The left and right Riemann–Liouville fractional derivatives of order $\alpha > 0$ for a given function $u(t)$, $t \in (a, b)$ are given as

$$ {}^{RL}\mathcal{D}^{\alpha}_{a,t}u(t) = \frac{d^n}{dt^n}\left[\mathcal{D}^{\alpha-n}_{a,t}u(t)\right] = \frac{1}{\Gamma(n-\alpha)}\frac{d^n}{dt^n}\int_a^t (t-\tau)^{n-\alpha-1}u(\tau)d\tau, \tag{1.3}$$

* Corresponding author.
E-mail addresses: kmowolabi@futa.edu.ng, owolabikm@ufs.ac.za (K.M. Owolabi).

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and

$$ {}^{RL}\mathcal{D}^{\alpha}_{b,t}u(t) = (-1)^n \frac{d^n}{dt^n}\left[\mathcal{D}^{\alpha-n}_{t,b}u(t)\right] = \frac{(-1)^n}{\Gamma(n-\alpha)}\frac{d^n}{dt^n}\int_t^b (\tau-t)^{n-\alpha-1}u(\tau)d\tau, \quad (1.4) $$

where $n>0$ is an integer satisfying $n-1\le\alpha<n$, and $\Gamma(\cdot)$ denotes the Euler's gamma function.

The Riesz fractional derivative of order $\alpha>0$ of a function $u(t)$, $t\in(a, b)$ is defined as

$$ {}^{Rz}\mathcal{D}^{\alpha}_{t}u(t) = c_\alpha\left({}^{RL}\mathcal{D}^{\alpha}_{a,t}u(t) + {}^{RL}\mathcal{D}^{\alpha}_{t,b}u(t)\right), \quad (1.5) $$

where $c_\alpha = -\frac{1}{2\cos(\alpha\pi/2)}$, $\alpha \neq 2s+1, s=0,1,2,\cdots$. ${}^{RL}\mathcal{D}^{\alpha}_{a,t}u(t)$ and ${}^{RL}\mathcal{D}^{\alpha}_{t,b}u(t)$ are the left and right Riemann–Liouville derivatives. Note that ${}^{Rz}\mathcal{D}^{\alpha}_{t}u(t)$ is ofter expressed as $\frac{\partial^\alpha u(t)}{\partial|t|^\alpha}$, see [18].

The left and right Caputo fractional derivatives of order $\alpha>0$ for a given function $u(t)$, $t\in(a, b)$ is respectively defined as

$$ {}^{C}\mathcal{D}^{\alpha}_{a,t}u(t) = \mathcal{D}^{\alpha-n}_{a,t}\left[u^{(n)}(t)\right] = \frac{1}{\Gamma(n-\alpha)}\int_a^t (t-\tau)^{n-\alpha-1}u^{(n)}(\tau)d\tau, \quad (1.6) $$

and

$$ {}^{C}\mathcal{D}^{\alpha}_{t,b}u(t) = \frac{(-1)^n}{\Gamma(n-\alpha)}\int_t^b (\tau-t)^{n-\alpha-1}u^{(n)}(\tau)d\tau, \quad (1.7) $$

for $n>0$ satisfying $n-1<\alpha\le n$.

The Caputo–Fabrizio fractional derivative of order $\alpha>0$ for function $u(t)$ can be defined as [5]

$$ {}^{CF}_{0}\mathcal{D}^{\alpha}_{t}y(t) = \frac{M(\alpha)}{1-\alpha}\int_0^t y'(\tau)\exp\left[-\frac{\alpha}{1-\alpha}(t-\tau)\right]d\tau, \quad (1.8) $$

where $M(\alpha)$ is a normalization function.

This work examines the major attractive characteristics which are the memory and chaos, in numerically simulating the dynamical systems. It should be mentioned that one of the major differences between the fractional-order and classical-order derivative equations is that, non-integer order models have memory; that is, the fractional-order system depends on the history of the model. Hence, we introduce and utilize the new nonlocal and nonsingular kernel version of the fractional derivative operator that was suggested by Atangana and Baleanu [1], in the sense of Caputo as

$$ {}^{ABC}_{0}\mathcal{D}^{\alpha}_{t}[u(t)] = \frac{M(\alpha)}{1-\alpha}\int_0^t u'(\tau)E_\alpha\left[-\alpha\frac{(t-\tau)^\alpha}{1-\alpha}\right]d\tau, \quad (1.9) $$

where $M(\alpha)$ has the same properties as in the case of the Caputo–Fabrizio fractional derivative. The one-parameter Mittag–Leffler function $E_\alpha(\cdot)$ is defined by the series expansion [1,31]

$$ E_\alpha(z) = \sum_{k=0}^{\infty}\frac{z^k}{\Gamma(\alpha k+1)}, \quad \alpha>0. \quad (1.10) $$

The rest of this paper is structured as follows. Numerical approximation schemes based on the fractional forward Euler method and the fraction Adams method is formulated in Section 2. In Section 3, we first introduce the dynamical system and deal with the local stability, global stability as well as the bifurcation analysis of the system. Numerical experiment for various value of α is given in Section 4, to examine the behaviour of such system with fractional derivatives. Conclusion is drawn in the last section.

2. Numerical methods for fractional differentiation

In this section, we introduce the numerical methods for the general initial-value problem of the form

$$ \mathcal{D}^{\alpha}_{0,t}u(t) = g(t,u(t)), \quad n-1<\alpha<n\in\mathbf{Z}^+, $$

$$ u^{(s)}(0) = u_0^s, \ s=0,1,2,\cdots,n-1. \quad (2.11) $$

where $\mathcal{D}^{\alpha}_{0,t}u(t)$ is expressed as the Caputo or Atangana–Baleanu fractional derivatives. The following existence and uniqueness results for Eqs. (2.11) have been established.

Theorem 2.1 (Existence result). *Let $\mathfrak{D} := [0,\omega]\times[u_0^0-\epsilon, u_0^0+\epsilon]$ with some $\omega>0$ and $\epsilon>0$. Assume that the function $g:\mathfrak{D}\to\mathbb{R}$ is continuous, we define $\omega := \min\{\omega, (\epsilon\Gamma(\alpha+1)/\|g\|_\infty)^{1/\alpha}\}$. Then, there exists a function $u:[0,\omega]\to\mathbb{R}$, for the solution of problem (2.11).*

Theorem 2.2 (uniqueness result). *Let $\mathfrak{D} := [0,\omega]\times[u_0^0-\epsilon, u_0^0+\epsilon]$ with $\omega>0$ and $\epsilon>0$. Assume $f:\mathfrak{D}\to\mathbb{R}$ is bounded on $\mathfrak{D}$ and satisfy a Lipschitz condition w.r.t. the another variable, that is*

$$ |g(t,x)-g(t,y)| \le M|x-y| \quad (2.12) $$

with $M>0$ being a constant independent of x, y and t. Then, there exists at most a function $u:[0,\omega^*]\to\mathbb{R}$ for the solution of problem (2.11)

Proof. The proofs of Theorems 2.11 and 2.2 using both the Caputo and the Atangana–Baleanu fractional operators can be found in [4,6] □

Given a subinterval $[t_k, t_{k+1}]$, for $k=0,1,2,\ldots,n-1$, the function $g(t)$ is approximated by a constant, that is

$$ g(t)\big|_{[t_k,t_{k+1}]} \approx \bar{g}(t)\big|_{[t_k,t_{k+1}]} = g(t_k) \quad (2.13) $$

we get

$$ \left[\mathcal{D}^{-\alpha}_{0,t}g(t)\right]_{t=t_n} = \frac{1}{\Gamma(\alpha)}\sum_{k=0}^{n-1}\int_{t_k}^{t_{k+1}}(t_n-\tau)^{\alpha-1}g(\tau)d\tau \approx \frac{1}{\Gamma(\alpha)}\sum_{k=0}^{n-1}\int_{t_k}^{t_{k+1}}(t_n-\tau)^{\alpha-1}g(t_k)d\tau \quad (2.14) $$

$$ = \sum_{k=0}^{n-1}\beta_{n-k-1}g(t_k), \quad (2.15) $$

where

$$ \beta_k = \frac{\Delta t^\alpha}{\Gamma(\alpha+1)}[(k+1)^\alpha - k^\alpha]. $$

Hence, we obtain the left fractional rectangular scheme

$$ \left[\mathcal{D}^{-\alpha}_{0,t}g(t)\right]_{t=t_n} \approx \sum_{k=0}^{n-1}\beta_{n-k-1}g(t_k). \quad (2.16) $$

Next, to derive the fractional forward Euler scheme, we approximate $\left[\mathcal{D}^{-\alpha}_{0,t}g(t)\right]$ at $t=t_{n+1}$ by using (2.16) to have

$$ u_{n+1} = \sum_{s=0}^{m-1}\frac{t_{n+1}^s}{s!}u_0^{(s)} + \Delta t^\alpha\sum_{s=0}^{n}\beta_{s,n+1}g(t_s,u_s), \quad (2.17) $$

where

$$ \beta_{s,n+1} = \frac{1}{\Gamma(\alpha+1)}[(n-s+1)^\alpha - (n-s)^\alpha]. $$

Again, to derive the fractional Adams method, we need to briefly discuss about the fractional trapezoidal formula. Considering subinterval $[t_k, t_{k+1}]$, we can approximate the function $u(t)$ by the piecewise polynomial of degree one as

$$ g(t)\big|_{[t_k,t_{k+1}]} \approx \bar{g}(t)\big|_{[t_k,t_{k+1}]} = \frac{t_{k+1}-t}{t_{k+1}-t_k}g(t_k) + \frac{t-t_k}{t_{k+1}-t_k}g(t_{k+1}), \quad (2.18) $$

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