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# Relationships among charges, fields, and potential on spherical surfaces boundary value problems

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ABSTRACT

This paper suggests a new point of view for the Poisson equation and its solution for the potential and field on the $d$ dimensional sphere, $S^d$, on which point charges are distributed. The available solutions for the potential on multidimensional spheres in the literature are purely mathematical, while the solution suggested here is motivated by physical intuition and requires minimal background; namely, basic laws of electrostatics and dimensional analysis. In this study, the modified Coulomb's law is presented by means of "dimensional reduction" and the use of equivalence between a point charge on $S^d$ and a charged ray in $\mathbb{R}^{d+1}$.

Besides formal detailed solutions and theorems, this paper presents concrete physical examples (unstudied or studied partly), such as distribution of charges/sources on a two-sphere; Dirichlet problem for currents on a truncated sphere; and fields and potentials created by "infinite" cones. Well-known statements about special charge distributions in Euclidean space must be reformulated and amended when dealing with the case of charges embedded in a spherical manifold.

## Introduction

The fundamental solution for the potential on spheres is related to a vast variety of physical potential problems, such as creeping flow in hydrodynamics (Goldstein and Eyal, 2018 [1]), currents (Eyal and Raz, 2016 [2]), electrostatics (Caillol, 2015 [3]), and geophysics (Stuart, 1967 [4]). Multidimensional problems are commonly used in particle and field theories, cosmological models, and also in field theory together with general relativity, e.g., the Kaluza-Klein theory. From time to time physicists build models that are embedded in compact manifolds, and the case of $S^d$ embedded in $R^{d+1}$ arises as a natural paradigm.

This paper follows the terminology of electrostatics: sources are called "charges" and denoted by $Q$ or $q$, the vector field is regarded as an electric field and denoted by $\overrightarrow{E}$, and the potential, $\varphi$, is defined by $\overrightarrow{E} = -\overrightarrow{\nabla}\varphi$. However, the relations that we obtain can be translated to other vector fields, such as electric current density, velocity field, and heat transfer.

The goal of this paper is to obtain the potential in $S^d$ that is created by a collection of embedded point charges on the sphere (fundamental solution). Martinez-Morales, 2005 [5] obtained this potential by using series of generalized Legendre polynomials. Bogomolov, 1977 [6] solved the potential function on $S^2$ for a distributed flow affected by vortices. Cohl, 2011 [7] considered the same problem in $S^d$ by a direct use of spherical coordinates in solving the Poisson equation locally. On the other hand, Crowdy, 2003 [8], and Caillol, 2015 [3] solved a modified version for the defining equation of Green's function in $S^2$ by adding a constant (a uniform charge density) to the Dirac delta function on its right hand side. Our approach for deriving the potential in $S^d$ that is created by a collection of point charges in it is based on a simple analog to an equivalent problem in the $\mathbb{R}^{d+1}$ space.

The reduction of the physical problem from $\mathbb{R}^{d+1}$ space to a lower dimensional manifold can follow from either of these reasons: (1) the setup is endowed with a symmetry (in our case symmetry under dilatation); or(2) the physical setup is constrained (as introduced by Goldstein and Eyal, 2018 [1] for a liquid in a narrow gap, so that fluctuations in the velocity field are averaged along the narrow gap coordinate, providing a two dimensional problem). Eyal and Raz, 2016 [2] presented a solution for the electric current in the particular case of $S^2$, based on a technique that is conceptually similar to the one presented in this paper for the general problem in $S^d$.

Section "Distribution of charges on the manifold $S^2$" deals with a two-sphere embedded in $\mathbb{R}^3$, by means of elementary naive and intuitive methods. Section "Generalization: distribution of charges on the manifold $S^d$" generalizes and mathematically formalizes the principles introduced in Section "Distribution of charges on the manifold $S^2$". Section "Closed polarized surface, charged $S^{d-1}$ embedded in $S^d$, and the average theorem" gives several insights about how geometric physical laws have to be amended when living in spheres. Section "Physical examples related to spheres" deals with physical examples, some of which are new, while some are for the sake of a consistency check with our results.

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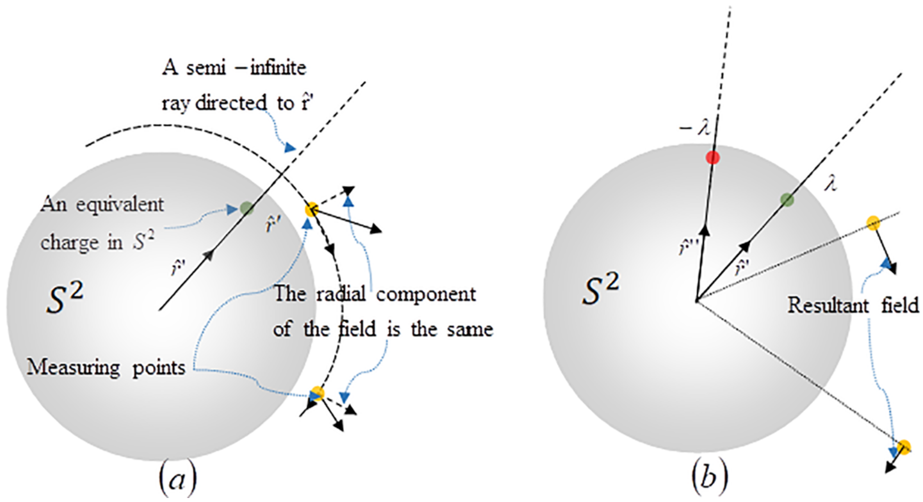


**Fig. 2.1.** A semi-infinite charged ray directed to $\hat{r}'$; (b) The cancelled radial field in $\mathbb{R}^3$ can be related to point charges on sphere $S^2$.

**Distribution of charges on the manifold $S^2$**

Given a charge distribution anywhere in $\mathbb{R}^3$ ($d = 2$) enables us to find the potential everywhere, by Coulomb's law integral, or mathematically, by convolution of the charge distribution with a Green's function in $\mathbb{R}^3$ (imposing zero potential at infinity). We take advantage of this property after showing that there is a simple and intuitive method that provides an equivalence between point charges on a sphere $S^2$ and a distribution of charges in $\mathbb{R}^3$. For this purpose, we consider a semi-infinite charged ray in $\mathbb{R}^3$, emanating from the origin in the $\hat{r}'$-direction (Fig. 2.1(a)), and let this $\hat{r}' = \hat{R}'$ be the direction to a point charge on a sphere (where the sphere is centered at the same origin). We show that for an appropriate distribution of charges along the ray with a sophistication, then the radial field is independent of angles.

Moreover, for oppositely charged rays (Fig. 2.1(b)), we will can eliminate the radial field at any arbitrary measuring point in $\mathbb{R}^3$. Due to this ability, the contribution of a single point charge in $S^2$ can be the same as that of a single charged ray in $\mathbb{R}^3$. This dimensional reduction, in which there are no field components that leave the sphere, enables us to simulate observables on the sphere.

Pictorially, considering a charge as a source of field-lines, every line emanating from a positive charge has no "infinity" to run to; so, on $S^2$, the field-line emanating from a positive charge has to enter into a negative charge, simply because there is no other possibility. This insight leads to an important requirement about charge distribution: on a sphere, the net charge must vanish.

The next section includes an example to illustrate this need. It also presents several results related to the field and potential created by a collection of rays in $\mathbb{R}^3$, as preparation to obtaining them on a sphere.

*Some results about charged rays*

The reduction of the 3*D* Euclidean problem into 2*D* on a sphere can achieved by the simple observation that the radial component of the field cancels, as we discuss in the following.

Due to the superposition principle, the electric field created by a uniformly charged ray with longitudinal density $\lambda$ has the form $\overrightarrow{E}(\overrightarrow{r}) = \lambda\overrightarrow{u}(\overrightarrow{r}, \textit{other lengths})$, where $\overrightarrow{u}$ is an unknown function. We now note that the units of the electric potential coincide with those of the charge density,[1] and units of the electric field are those of the charge density divided by length. Armed with this insight and provided that the physical problem does not contain any intrinsic length, the field should be $\overrightarrow{E} = \frac{\lambda}{r}\overrightarrow{h}(\hat{r})$, where *h* is some function that does not depend on the distance to the origin.

Moreover, we now show that the radial component of the field is independent of angles. Let us consider the uniformly charged-ray depicted in Fig. 2.2(a) with a density $\lambda$. If one considers and the arcs ($\widehat{BC}$ and $\widehat{BC}$), centered at the origin, shown in Fig. 2.2(b); since the field is conservative, we have:

$$\int_A^B \overrightarrow{E}(\overrightarrow{r})\cdot d\overrightarrow{r} + \int_B^C \overrightarrow{E}(\overrightarrow{r})\cdot d\overrightarrow{r} + \int_C^D \overrightarrow{E}(\overrightarrow{r})\cdot d\overrightarrow{r} + \int_D^A \overrightarrow{E}(\overrightarrow{r})\cdot d\overrightarrow{r} = 0 \tag{2.1}$$

Since the field is proportional to $\frac{1}{r}$ and arc elementary length is proportional to $rd\theta$, then contributions from arcs $\widehat{BC}$ and $\widehat{BC}$ with the same angles cancel each other, and Eq. (2.1) becomes:

$$\int_A^B \overrightarrow{E}(\overrightarrow{r})\cdot d\overrightarrow{r} + \int_C^D \overrightarrow{E}(\overrightarrow{r})\cdot d\overrightarrow{r} = 0 \tag{2.2}$$

Eq. (2.2) shows that the integral along the segment $\overline{AB}$ is cancelled by the integral along the segment $\overline{CD}$, meaning that $E_{rA} = E_{rD}$ namely, the radial component of the field is independent of the angles.

Can we leverage the above property of the field? The answer is yes. If one considers a positively charged ray together with a negatively charged ray, then the resultant field is tangential, with no radial component at all, as shown in Fig. 2.1b. More generally, when we consider a collection of rays with densities whose sum is zero, the resultant field is tangential. The very meaning of this fact is "life on a sphere".

Concrete examples for which the field created by sources lives only on a sphere are current sources entering a (poor) conductor (Eyal and Raz, 2016 [2]) and creeping flow in a narrow gap between two concentric spheres (Goldstein and Eyal, 2018 [1]), generated by several sources.

*Calculation of the potential and field created by a charged ray in $\mathbb{R}^3$*

As discussed above, we now show that the potential on the sphere created by point charges can be evaluated equivalently as the potential created by uniformly charged rays in $\mathbb{R}^3$.

Let a ray be parametrized as: $\overrightarrow{r}' = \hat{r}'s$, $0 < s < \infty$. Using the superposition principle, the potential at point $\overrightarrow{r}$ should look like this:[2]

$$\varphi(\overrightarrow{r}) = \frac{1}{4\pi}\int dq\frac{1}{\|\overrightarrow{r}-\overrightarrow{r}'\|} = \frac{1}{4\pi}\int_0^\infty ds\lambda\frac{1}{(r^2+s^2-2rs(\hat{r}\cdot\hat{r}'))^{1/2}} \tag{2.3}$$

[1] $-\overrightarrow{\nabla}\varphi = \overrightarrow{E} \Rightarrow [E] = \frac{[\varphi]}{[length]}$, from Gauss: $E^{*}area = charge \Rightarrow [\varphi] = \frac{[charge]}{[length]}$, like $\lambda = \frac{d(charge)}{d(length)}$.

[2] $\frac{1}{\epsilon_0}$ is included in the charges