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Improvement in accuracy for dimensionality reduction and reconstruction of noisy signals. Part II: The case of signal samples

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ABSTRACT

In this paper, a novel interpretation of the problem of dimensionality reduction and reconstruction of random signals is studied. The problem and its solution target highly noisy signals and are considered in terms of signal samples. The solution is given by an iteration procedure where, on each iteration, solution parameters are optimally determined from the minimization of an associated cost function. The associated error diminishes with the increase in the number of iterations. The advantages of the considered technique are discussed and illustrated numerically.

1. Introduction

1.1. Motivation and statement of problem

The preceding paper [1] addresses the optimal solution of the new problem of dimensionality reduction and reconstruction of random signals in terms of random vectors. Here, we provide and study a numerical technique for the experimental testing of the method proposed in [1]. The proposed approach targets the case of highly noisy signals. Since for the experimental testing, in the training stage or for numerical analysis, signals are replaced with their samples, we here consider an interpretation of the problem in [1] and its solution in terms of signal samples. This interpretation implies a new method for the optimal determination of the so called injection[1] $\mathbf{v}$ (see (15) below) and a new technique of its justification (see Sections 2.2 and 2.3 that follow). As a result, the solution of the related problem (provided in Section 2 below) has special associated advantages. In particular, the new procedure allows us to iteratively increase the associated accuracy. This technique was not considered in [1].

Further, the proposed technique is applicable to a version of the blind identification problem studied in [2,3]. This issue is considered in more detail in Section 1.2 below. This is also a motivation to develop the approach presented in the Sections that follow.

The problem under consideration is as follows.

Let $X\in\mathbb{R}^{m\times s}$, $Y\in\mathbb{R}^{n\times s}$ and $V\in\mathbb{R}^{q\times s}$ be samples of source signals $\mathbf{x}$, noisy observed signal $\mathbf{y}$ and injection $\mathbf{v}$ [1], respectively, where s is a number of samples. Signals $\mathbf{x}$, $\mathbf{y}$ and $\mathbf{v}$ are random. We denote $k\le\min\{m, n\}$, $k=k_1+k_2$ where k_1 and k_2 are nonnegative integers, and write $\|\cdot\|$ for the Frobenius norm.

Further, let Q be a transformation of samples Y and V in matrix Z, i.e.,

$$Z=Q(Y,V).$$

Denote $W=[Y^TZ^T]^T$.

Problem: Find V, $H_1\in\mathbb{R}^{m\times k_1}$, $H_2\in\mathbb{R}^{k_1\times n}$, $H_3\in\mathbb{R}^{m\times k_2}$, $H_4\in\mathbb{R}^{k_2\times q}$ that solve

$$\min_{V}\min_{H_1,H_2}\min_{H_3,H_4}\|X-[H_1H_2Y+H_3H_4Q(Y,V)]\|^2, \tag{1}$$

and Q which implies

$$WW^T=\begin{bmatrix}YY^T & \mathbb{O}\\ \mathbb{O} & ZZ^T\end{bmatrix}. \tag{2}$$

Recall that $c=k/\min\{m,n\}$ stands for the reduction ratio [1] where $k=k_1+k_2$ and $k\le\min\{m, n\}$.

In experimental testing, signals are modeled. The commonly used model of $\mathbf{y}$ (see, e.g., Brillinger and co-authors [4–7]) is given by $\mathbf{y}=A\mathbf{x}+\xi$ or $\mathbf{y}=\mathbf{x}+\xi$ where A is a known matrix and ξ denotes a modeled additive noise. Example 2 below and the examples in [1] use this model of $\mathbf{y}$. We note that although in general,

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[1] This term has been introduced in [1].

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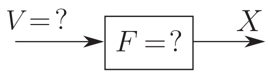

Fig. 1. Block-diagram of the blind identification problem.

the source signal **x** is not available, for the modeled signal **y** under the assumption that, in particular, A is invertible, the sample matrix X can be represented in terms of matrices Y, A and sample noise matrix N. Therefore, here, since we consider the case of experimental testing, we adopt the assumption that X is available, similar to the work in [4–7].

Further, transform F given by

$$F(Y,V) = H_1H_2Y + H_3H_4Q(Y,V), \quad (3)$$

where V, H_1, H_2, H_3, H_4 and Q solve problem (1) is called the double reduced-rank transform (DRRT2). This is because matrices $F_1 = H_1H_2$ and $F_2 = H_3H_4$ are of ranks k_1 and k_2, respectively. In (3), matrices H_2 and H_4 provide the signal dimensionality reduction, and matrices H_1 and H_3 provide the signal reconstruction. Condition (2) allows us to reduce the computational complexity of DRRT2. This issue was detailed in [1].

1.2. Novelties

The DRRT2 structure allows us to combine an optimal determination of injection sample V with the optimal determination of matrices H_1, H_2, H_3, H_4. As a result, for DRRT2, the technique of the optimal determination of V is different from the method of finding optimal injection **v** in [1]; it is defined as an iterative procedure detailed in Section 2.1 below. The associated theoretical justification is presented in Section 2. Note that DRRT2 contains five matrices to optimize, V, H_1, H_2, H_3, H_4, which enable improvement in the transform performance.

As mentioned in Section 2.3 of [1], the problem under consideration can also be interpreted as a system identification problem [8,9]. Moreover, problem (1)–(2) that causes DRRT2 can also be interpreted as a new formulation of the blind identification problem considered, in particular, in [2,3]. In Fig 1, the corresponding block-diagram is represented. By the approach in [2], the problem in (1) is interpreted differently than that in Section 1.1 above, so that X is an available output of the system, F is an unknown system function, V is an unknown input and Y is a known 'auxiliary' signal. Further differences from Abed-Meriam et al. [2] are that in (1), the unknown system function F is formed by four matrices H_1, H_2, H_3, H_4, i.e., contains more unknowns than in Abed-Meriam et al. [2], and moreover, matrices H_1, H_2, H_3, H_4 have special sizes. Therefore, the problem in (1), in terms of matrices $F_1 = H_1H_2$ and $F_2 = H_3H_4$, can be reformulated as

$$\min_{V}\min_{F_1}\min_{F_2} \|X - [F_1Y + F_2Q(Y,V)]\|^2, \quad (4)$$

subject to

$$\operatorname{rank} F_1 \le k_1 \quad \text{and} \quad \operatorname{rank} F_2 \le k_2. \quad (5)$$

By the above reasons, the method in [2] cannot be applied here. Therefore, below in Section 2, we propose a new method for a solution of the problem in (1)–(2) (and, as a result, in (4)–(5)). On the above basis, we believe that the proposed technique is worthwhile in its own right.

The corresponding block-diagram is represented in Fig. 2 where $\widetilde{X}$ is an estimate of X obtained from the solution of the problem in (4) and (5).

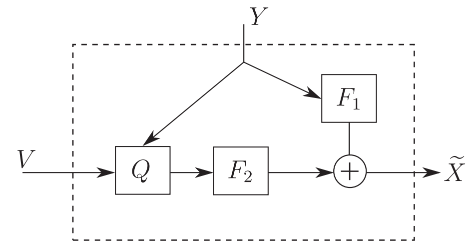

Fig. 2. Illustrations to the problem given by (3)–(5).

2. Determination of DRRT2

2.1. Solution of problem in (1), (2)

First, in (1), we determine $Q(Y, V)$. The pseudo-inverse matrix of matrix M is denoted by $M^\dagger$.

Theorem 1. *Let*

$$Q(Y,V) = Z = VG, \quad (6)$$

where $G = I - Y^\dagger Y$ *. Then condition* (2) *is true.*

Proof. We observe that $Y^\dagger Y = Y^T(Y^\dagger)^T$ [10]. Then

$$YZ^T = YG^TV^T = Y(I - Y^\dagger Y)^T V^T$$
$$= (Y - YY^\dagger Y)V^T = (Y - Y)V^T = \mathbb{O}$$

while

$$ZZ^T = V(I - Y^\dagger Y)(I - Y^\dagger Y)^T V^T \neq \mathbb{O}.$$

Therefore, $WW^T = \begin{bmatrix} YY^T & YZ^T \\ ZY^T & ZZ^T \end{bmatrix} = \begin{bmatrix} YY^T & \mathbb{O} \\ \mathbb{O} & ZZ^T \end{bmatrix}$. □

As a result, (2) and (6) imply that the problem in (1) can now be written as

$$\min_{V}\min_{H_1,H_2}\min_{H_3,H_4} \|X - [H_1H_2Y + H_3H_4Z]\|^2$$
$$= \min_{H_1,H_2} \|X - H_1H_2Y\|^2 + \min_{V}\min_{H_3,H_4} \|X - H_3H_4Z\|^2 - \|X\|^2. \quad (7)$$

Let us denote

$$S_Y = XY^T(YY^T)^{1/2\dagger} \quad \text{and} \quad T_Y = S_YS_Y^T = XY^TY^\dagger X^T,$$

and write $U_{T_Y}\Sigma_{T_Y}U_{T_Y}^T = T_Y$ for the SVD of T_Y.

Definition 1. Matrix V is called a well-defined sample injection if

$$VG \neq \mathbb{O}. \quad (8)$$

Otherwise, V is called an ill-defined sample injection.

The null-space of matrix M is denoted by $\mathcal{N}(M)$.

Definition 2. Let $Y = [y_1, \ldots, y_s]$ and $V = [v_1, \ldots, v_s]$ where y_j and v_j are columns of Y and V, respectively, for $j = 1, \ldots, s$. Let $F_1 \in \mathbb{R}^{m\times n}$ and $F_2 \in \mathbb{R}^{m\times q}$. Sample matrices Y and V are called jointly independent if the equation

$$F_1Y + F_2V = \mathbb{O} \quad (9)$$

can only be satisfied if $y_j \in \mathcal{N}(F_1)$ and $v_j \in \mathcal{N}(F_2)$, for $j = 1, \ldots, s$.

On the basis of representation (7), the proposed method for solution of problem in (1) consists of the following iterative steps, where V is assumed to be the well-defined sample injection and matrices Y and V are jointly independent.

Step 1: For an arbitrary $V = V^{(0)}$, solve

$$\min_{H_1,H_2}\min_{H_3,H_4} \|X - [H_1H_2Y + H_3H_4Z^{(0)}]\|^2, \quad (10)$$