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# Domain decomposition modified with characteristic mixed finite element of compressible oil-water seepage displacement and its numerical analysis☆

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ABSTRACT

A parallel algorithm, structured by domain decomposition and characteristic mixed finite element, is presented for solving three-dimensional displacement problem of compressible seepage flow. Decomposing the computational domain into several subdomains, we define a characteristic function to approximate the value on interior boundary at previous time level and obtain numerical solutions implicitly in subdomains in parallel. The flow equation is treated by the method of mixed finite element and the saturation equation is approximated by the method of characteristic finite element. For a model problem we apply variation form, domain decomposition, the method of characteristics, the principle of energy, negative norm estimates, induction hypothesis, the theory and technique of priori estimates of differential equations to derive optimal error estimate in $l^2$ norm. Numerical data are consistent with theoretical analysis and show that this method is effective in solving actual applications. Then it can solve the well-known problem.



## 1. Introduction

For the black-oil model, most important in modern numerical simulation of oil reservoir, the compressibility and three-phase (oil, water and oil) displacement should be considered together to avoid distorted approximation. For convenience to structure the computational algorithm, based on the theories of mathematics and mechanics, the rigorous numerical analysis should be illustrated. In general, model problem is discussed, so we consider the displacement problem of two-phase (oil and water) flow. High-pressure pumps are used to inject water into under-ground geologic formations, and crude oil is retrieved from the underground deposit at production wells. This technique is popular and important in modern oil displacement. The displacement of two phase is interpreted by the fact that injected water displaces crude oil in reservoir and crude oil is produced. For compressible miscible displacement the density actually not only depends on the pressure

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but also depends on the saturation. Though the mathematical model is formulated, it is very hard to discuss its numerical analysis such as numerical method and theoretical analysis.

The mathematical model is formulated by a nonlinear system of three partial differential equations with initial-boundary value conditions [1–5]:

$$\phi\frac{\partial\rho}{\partial t} = -\nabla\cdot\mathbf{u} + q,\ \ (X,t)\in\Omega\times J,\ J=(0,T], \tag{1.1a}$$

$$\mathbf{u} = -\frac{\kappa}{r}\nabla p,\ (X,t)\in\Omega\times J, \tag{1.1b}$$

$$\phi\frac{\partial(\rho c)}{\partial t} = -\nabla\cdot(c\mathbf{u}) + \nabla\cdot(D\nabla c) + q\tilde{c},\ \ (X,t)\in\Omega\times J, \tag{1.2}$$

where $\phi=\phi(X)$ is the porosity. $\rho$ is the density, defined by a function of the pressure $p$ and the saturation $c$,

$$\rho = \rho(c,p) = \rho_0(c)[1+\alpha_0(c)p]. \tag{1.3a}$$

$\rho_0$ denotes the density in the normal state, and it is defined by a linear function of the densities of original fluid, $\rho_r$, and injected fluid, $\rho_i$,

$$\rho_0(c) = (1-c)\rho_r + c\rho_i, \tag{1.3b}$$

where $\rho_r$ and $\rho_i$ are positive constants.

The compressibility factor, $\alpha_0(c)$, is defined by a linear combination of two positive constants, $\alpha_r$ and $\alpha_i$,

$$\alpha_0(c) = (1-c)\alpha_r + c\alpha_i, \tag{1.3c}$$

where $\alpha_r$ and $\alpha_i$ denote the compressibility factors of original fluid and injected fluid, respectively. The viscosity, $\mu=\mu(c)$, is defined by

$$\mu(c) = \left((1-c)\mu_r^{1/4} + c\mu_i^{1/4}\right)^4, \tag{1.4}$$

where $\mu_r$ and $\mu_i$ denote the viscosities of different fluids.

The viscosity of mixed fluid $r$ is defined by

$$r(c,p) = \frac{\mu(c)}{\rho(c,p)}. \tag{1.5}$$

$\mathbf{u}=\mathbf{u}(X,t)$ is Darcy velocity, and $\kappa=\kappa(X)$ is the permeability. $\Omega$ is a bounded domain of $R^3$ and its boundary is denoted by $\partial\Omega$. $q(X,t)$ is the production. The diffusion coefficient, $D=D(X)$, is determined by Fick Law. $\tilde{c}(X,t)$ is assigned by the number 1 at injection well ($q>0$), and is equal to $c$ at production well ($q<0$). Using the definition of $\rho(c,p)$, we rewrite (1.1a) as follows

$$\phi\rho_0(c)\alpha_0(c)\frac{\partial p}{\partial t} + \phi\{(\rho_i-\rho_r)[1+\alpha_0(c)p] + (\alpha_i-\alpha_r)\rho_0(c)p\}\frac{\partial c}{\partial t} - \nabla\cdot\left(\tfrac{\kappa}{r}\nabla p\right) = q,\ \ (X,t)\in\Omega\times J. \tag{1.6}$$

Using $\phi\frac{\partial(c\rho)}{\partial t} = \phi(c\frac{\partial\rho}{\partial t} + \rho\frac{\partial c}{\partial t})$, and combining (1.2) with (1.1a), we have

$$\phi\rho\frac{\partial c}{\partial t} + \mathbf{u}\cdot\nabla c - \nabla\cdot(D\nabla c) = q(\tilde{c}-c),\ \ (X,t)\in\Omega\times J. \tag{1.7}$$

Suppose that the fluid is not permeable at $\partial\Omega$, that is to say that the following conditions hold

$$\begin{aligned}&\mathbf{u}\cdot\sigma = 0,\ \ (X,t)\in\partial\Omega\times J,\\ &D\nabla c\cdot\sigma = 0,\ \ (X,t)\in\partial\Omega\times J.\end{aligned} \tag{1.8}$$

Initial condition is defined by

$$p(X,0) = p_0(X),\ X\in\Omega,\quad c(X,0) = c_0(X),\ X\in\Omega. \tag{1.9}$$

For incompressible two-phase seepage displacement in $R^2$, Douglas and Ewing et. al. put forward characteristic finite difference and characteristic finite element under periodic assumption and gives rigorous convergence analysis [6–10]. Combining normal finite difference or finite element with the method of characteristics, they present two different composite schemes. The schemes can reflect the first-order hyperbolic nature of convection–diffusion equation, decrease truncation error, overcome numerical oscillation and dispersion and can improve the stability and accuracy greatly. The compressibility must be considered in simulating the modern enhanced oil displacement [4,5,11]. Introducing periodic assumption, Douglas and Yuan firstly discuss characteristic finite element and characteristic mixed finite element, obtain optimal order error estimate in $L^2$ norm and give a powerful tool for solving the challenging benchmark problem [11–14]. In numerical simulation of modern petroleum exploration and development, many factors such as the large-scale computation, three-dimensional domain and a long time interval are involved. The number of nodes maybe amount up to tens of thousands or millions, so traditional methods cannot solve this problem well. Therefore, some new modern parallel computation methods