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Symmetric Grothendieck polynomials, skew Cauchy identities, and dual filtered Young graphs

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ABSTRACT

Symmetric Grothendieck polynomials are analogues of Schur polynomials in the K-theory of Grassmannians. We build dual families of symmetric Grothendieck polynomials using Schur operators. With this approach we prove skew Cauchy identity and then derive various applications: skew Pieri rules, dual filtrations of Young's lattice, generating series and enumerative identities. We also give a new explanation of the finite expansion property for products of Grothendieck polynomials.

1. Introduction

Symmetric Grothendieck polynomials, also known as stable Grothendieck polynomials, are certain K-theoretic deformations of Schur functions. These functions were first studied by Fomin and Kirillov [8] as a stable limit of more general Grothendieck polynomials that generalize Schubert polynomials in another direction.

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The symmetric Grothendieck polynomial G_λ can be defined by the following combinatorial formula due to Buch [4]

$$G_\lambda(x_1, x_2, \ldots) = \sum_T (-1)^{|T|-|\lambda|} \prod_{i \geq 1} x_i^{\#i\text{'s in } T},$$

where the sum runs over shape λ *set-valued tableaux* T, a generalization of semistandard Young tableaux so that boxes contain sets of integers.

Being a generalization of the Schur basis, symmetric Grothendieck polynomials share with it many similarities. However $\{G_\lambda\}$ is *inhomogeneous* and of *unbounded* degree when defined for infinitely many variables $(x_1, x_2, \ldots)$, i.e., it is an element of the completion $\hat{\Lambda}$ of the ring Λ of symmetric functions. For example, $G_{(1)} = e_1 - e_2 + e_3 - \cdots$, where e_k is the kth elementary symmetric function. It is thus surprising that $\{G_\lambda\}$ is a *basis* of a certain ring: each product $G_\mu G_\nu$ is a *finite* linear combination of $\{G_\lambda\}$. When μ is a single row or column partition, finite expansion was a consequence of Pieri rules proved by Lenart [13]. In a general case, this property follows from a Littlewood–Richardson rule given by Buch [4]. The ring spanned by Grothendieck polynomials is related to the K-theory of Grassmannians [4]. There is also an important basis $\{g_\lambda\}$ of Λ, dual to $\{G_\lambda\}$, that was described via plane partitions and studied by Lam and Pylyavskyy [11].

In this paper we study symmetric *skew* Grothendieck polynomials via noncommutative *Schur operators*. We used these operators in [22] to prove dualities for certain two-parameter deformations of Grothendieck polynomials. Employing classical Schur operators turns out to be beneficial for obtaining a number of new properties.

Our main results are the following.

1.1. Skew Cauchy identity

We prove the following identity that becomes our central object.

Theorem 1.1. *Let μ, ν be any fixed partitions, then*[1]

$$\sum_\lambda G_{\lambda /\!\!/ \mu}(x_1, x_2, \ldots) g_{\lambda/\nu}(y_1, y_2, \ldots) = \prod_{i,j} \frac{1}{1 - x_i y_j} \sum_\kappa G_{\nu /\!\!/ \kappa}(x_1, x_2, \ldots) g_{\mu/\kappa}(y_1, y_2, \ldots).$$

We give a number of applications of this identity using it in both operator and generating function forms. Our approach is based on Schur operators as in Fomin [6]. For Schur functions such an identity was given by Zelevinsky in the Russian translation of Macdonald's book [14]. Macdonald [14, Ch.1 notes] mentioned that this result has apparently been discovered independently by Lascoux, Towber, Stanley, Zelevinsky. It is also known for analogues, e.g., shifted Schur functions [6,17]. Borodin's symmetric functions [2] generalizing Hall–Littlewood polynomials, also satisfy Cauchy identities which

[1] The shape $\lambda /\!\!/ \mu$ is not the usual skew shape λ/μ.