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The application of representation theory in directed strongly regular graphs

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ABSTRACT

The concept of directed strongly regular graphs (DSRG) was introduced by Duval in 1988 [3]. In the present paper, we use representation theory of finite groups in order to investigate the directed strongly regular Cayley graphs. We first show that a Cayley graph $\mathcal{C}(G,S)$ is not a directed strongly regular graph if S is a union of some conjugate classes of G. This generalizes an earlier result of Leif K. Jørgensen [7] on abelian groups. Secondly, by using induced representations, we have a look at the Cayley graph $\mathcal{C}(N \rtimes_\theta H, N_1 \times H_1)$ with $N_1 \subseteq N$ and $H_1 \subseteq H$, determining its characteristic polynomial and its minimal polynomial. Based on this result, we generalize the semidirect product method of Art M. Duval and Dmitri Iourinski in [4] and obtain a larger family of directed strongly regular graphs. Finally, we construct some directed strongly regular Cayley graphs on dihedral groups, which partially generalize the earlier results of Mikhail Klin, Akihiro Munemasa, Mikhail Muzychuk, and Paul Hermann Zieschang in [8]. By using character theory, we also give the characterization of directed strongly regular Cayley graphs $\mathcal{C}(D_n, X \cup Xa)$ with $X \cap X^{(-1)} = \emptyset$.

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1. Introduction

A *directed strongly regular graph* (DSRG, since it will appear many times, it will be abbreviated as "DSRG" in the following) with parameters (n, k, μ, λ, t) is a k-regular directed graph on n vertices such that every vertex is on t 2-cycles, and the number of paths of length two from a vertex x to a vertex y is λ if there is an arc from x to y and it is μ otherwise. A DSRG with $t = k$ is an (undirected) strongly regular graph (SRG). Duval showed that a DSRG with $t = 0$ is a doubly regular tournament. Therefore, it is usually assumed that $0 < t < k$.

Another definition of a directed strongly regular graph can be given in terms of adjacency matrices. Let D be a directed graph with n vertices. Let $A = \mathbf{A}(D)$ denote the adjacency matrix of D; $I = I_n$ denote the $n \times n$ identity matrix; $J = J_n$ denote the all-ones matrix. Then D is a DSRG with parameters (n, k, μ, λ, t) if and only if (i) $JA = AJ = kJ$ and (ii) $A^2 = tI + \lambda A + \mu(J - I - A)$. Duval [3] developed necessary conditions on the parameters of (n, k, μ, λ, t)-DSRG and calculated the spectrum of a DSRG.

Proposition 1.1 *([3]). A DSRG with parameters (n, k, μ, λ, t) and $0 < t < k$ satisfy*

$$k(k + (\mu - \lambda)) = t + (n - 1)\,\mu, \tag{1.1}$$

$$d^2 = (\mu - \lambda)^2 + 4\,(t - \mu)\,, d | 2k - (\lambda - \mu)(n - 1),$$

$$\frac{2k - (\lambda - \mu)(n - 1)}{d} \equiv n - 1 (\text{mod } 2), \left| \frac{2k - (\lambda - \mu)(n - 1)}{d} \right| \leqslant n - 1,$$

where d is a positive integer, and

$$0 \leqslant \lambda < t, 0 < \mu \leqslant t, -2\,(k - t - 1) \leqslant \mu - \lambda \leqslant 2\,(k - t)\,. \tag{1.2}$$

Proposition 1.2 *([3]). A DSRG with parameters (n, k, μ, λ, t) has three distinct integer eigenvalues*

$$k > \rho = \frac{1}{2}\,(-\,(\mu - \lambda) + d) > \sigma = \frac{1}{2}\,(-\,(\mu - \lambda) - d)\,. \tag{1.3}$$

The multiplicities are

$$1,\; m_\rho = -\frac{k + \sigma\,(n - 1)}{\rho - \sigma},\; m_\sigma = \frac{k + \rho\,(n - 1)}{\rho - \sigma} \tag{1.4}$$

respectively.

Proposition 1.3. *([3]) If D is a DSRG with parameters (n, k, μ, λ, t), then its complement D' is also a DSRG with parameters $(n', k', \mu', \lambda', t')$, where $k' = (n - 2k) + (k - 1)$, $\lambda' = (n - 2k) + (\mu - 2)$, $t' = (n - 2k) + (t - 1)$ and $\mu' = (n - 2k) + \lambda$.*