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Characterising bounded expansion by neighbourhood complexity

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ABSTRACT

We show that a graph class $\mathcal{G}$ has *bounded expansion* if and only if it has bounded *r-neighbourhood complexity*, i.e., for any vertex set X of any subgraph H of any $G \in \mathcal{G}$, the number of subsets of X which are exact *r*-neighbourhoods of vertices of H on X is linear in the size of X . This is established by bounding the *r*-neighbourhood complexity of a graph in terms of both its *r*-centred colouring number and its weak *r*-colouring number, which provide known characterisations to the property of bounded expansion.

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1. Introduction

Graph classes of *bounded expansion* (and their further generalisation, nowhere dense classes) have been introduced by Nešetřil and Ossona de Mendez [24–26] as a general model of *structurally sparse* graph classes. They include and generalise many other natural sparse graph classes, among them all classes of bounded degree, classes of bounded genus, and classes defined by excluded (topological) minors. Nowhere dense classes even include classes that locally exclude a minor, which in turn generalises graphs with locally bounded treewidth.

The appeal of this notion and its applications stems from the fact that bounded expansion has turned out to be a very robust property of graph classes with various seemingly unrelated characterisations (see [19,26]). These include characterisations through the density of shallow minors [24],

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quasi-wideness [3], low treedepth colourings [24], and generalised colouring numbers [32]. The latter two are particularly relevant towards algorithmic applications, as we will discuss in the sequel. Furthermore, there is good evidence that real-world graphs (often dubbed ‘complex networks’) might exhibit this notion of structural sparseness [6,28], whereas stricter notions (planar, bounded degree, excluded (topological) minors, etc.) do not apply.

It seems unlikely that bounded expansion and nowhere dense classes admit global Robertson–Seymour style decompositions as they are available for classes excluding a fixed minor [29], a topological minor [21], an immersion [31], or an odd minor [5]. However, Nešetřil and Ossona de Mendez showed [25] that bounded expansion and nowhere dense classes admit a ‘local’ decomposition, a so-called *low r -treedepth colouring*, in the following sense: there exists a function $f : \mathbb{N} \rightarrow \mathbb{N}$ (depending on the graph class) such that for every integer r , every graph G from a bounded expansion (nowhere dense) class can be coloured with $f(r)$ (respectively $f(r)n^{o(1)}$) colours such that every union of $p < r$ colour classes induces a graph of treedepth at most p . We denote by $\chi_r(G)$ the minimal number of colours needed for an r -treedepth colouring of G . These types of colourings generalise the star-colouring number ([26], Section 7.1) introduced by Fertin, Raspaud, and Reed [12]. In that context, low r -treedepth colourings are usually called *r -centred colourings*³ (the precise definition of which we defer to Section 2).

This ‘decomposition by colouring’ has direct algorithmic implications. For example, counting how often an h -vertex graph appears in a host graph G as a subgraph, induced subgraph or homomorphism is possible in linear time [25] through the application of low r -centred (r -treedepth) colourings. A more precise bound of running time $O(|c(G)|^{2h} 6^h h^2 \cdot |G|)$ for all three problems was shown by Demaine et al. [6] if an appropriate low treedepth colouring c is provided as input. Low r -centred (r -treedepth) colourings can be further used to check whether an existential first-order sentence is true [26] or to approximate the problems $\mathcal{F}$ -DELETION and INDUCED- $\mathcal{F}$ -DELETION to within a factor that only depends on the expansion of the class $\mathcal{G}$ the graph G belongs to and on the forbidden set $\mathcal{F}$ [28].

Another characterisation of bounded expansion is obtained via the *weak r -colouring numbers*, denoted by $\text{wcol}_r(G)$. The name ‘colouring number’ reflects the fact that the weak 1-colouring number is sometimes also called the *colouring number* of the graph, which is one more than the *degeneracy* of the graph. Roughly, the weak colouring number describes how well the vertices of a graph can be linearly ordered such that for any vertex v , the number of vertices that can reach v via short paths that use higher-order vertices is bounded. We postpone the precise definition of weak r -colouring numbers to Section 2, but let us emphasise their utility: Grohe, Kreutzer, and Siebertz [20] used weak r -colouring numbers to prove the milestone result that first-order formulas can be decided in almost linear time for nowhere-dense classes (improving upon a result by Dvořák, Král, and Thomas for bounded expansion classes [11] and the preceding work for smaller sparse classes [4,13,17,30]).

Our work here centres on a new characterisation, motivated by recent progress in the area of kernelisation. This field, a subset of parametrised complexity theory, formalises polynomial-time pre-processing of computationally hard problems. For an introduction to kernelisation we refer the reader to the seminal work by Downey and Fellows [8]. Gajarský et al. [18] extended the meta-kernelisation framework initiated by Bodlaender et al. [2] for bounded-genus graphs to nowhere-dense classes (notable intermediate results were previously obtained for excluded-minor classes [14] and classes excluding a topological minor [23]). In a largely independent line of research, Drange et al. [9] recently provided a kernel for DOMINATING SET on nowhere-dense classes. Previous results showed kernels for planar graphs [1], bounded-genus graphs [2], apex-minor-free graphs [14], graphs excluding a minor [15] and graphs excluding a topological minor [16].

A feature exploited heavily in the above kernelisation results for bounded expansion classes is that for any graph G from such a class, every subset $X \subseteq G$ has the property that the number of ways vertices of G connect to X is linear in the size of X . Formally, we have that

$$| \{N(v) \cap X\}_{v \in V(G)} | \leq c \cdot |X|$$

³ Depending on the way r -treedepth colourings are defined, r -centred colourings might appear in the literature as $r - 1$ -treedepth colourings, as for example in [26]. For convenience, here we define them in a way so that the gap in the depth r is alleviated.

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