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Length enumeration of fully commutative elements in finite and affine Coxeter groups

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An element w of a Coxeter group W is said to be *fully commutative* if any reduced expression of w can be obtained from any other by a sequence of transpositions of adjacent commuting generators. These elements were described in 1996 by Stembridge in the case of finite irreducible groups, and more recently by Biagioli, Jouhet and Nadeau (BJN) in the affine cases. We focus here on the length enumeration of these elements. Using a recursive description, BJN established systems of non-linear q-equations for the associated generating functions. Here, we show that an alternative recursive description leads to explicit expressions for these generating functions.

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1. Introduction

Let (W, S) be a Coxeter system. An element w of W is said to be *fully commutative* (or fc for short) if any reduced expression of w can be obtained from any other by a sequence of transpositions of adjacent commuting generators. For instance, in the finite or affine symmetric group, fc elements coincide with 321-avoiding permutations [7,22]. The description and enumeration of fully commutative elements has been of interest in the algebraic combinatorics literature for about 20 years, starting with the work of Stembridge [31–33]. We refer to the above papers and to [5] for motivations of this topic. Stembridge classified Coxeter groups having finitely many fc elements, and was able to count those elements in each case [31,33].

More recently, several authors got interested, not only in the number of such elements, but also in their q-enumeration, where the variable q records their Coxeter length [5,23]. For instance, in the symmetric group A_2, all elements except the maximal permutation are fc, and their length enumeration yields the polynomial

$$A_2^{FC}(q) = 1 + 2q + 2q^2.$$

This point of view naturally extends the study to arbitrary Coxeter groups W (having possibly infinitely many fc elements), since they still have finitely many elements of given length. In this case the length enumeration of fc elements in W gives rise to a power series rather than a polynomial. We denote this series by $W^{FC}(q)$.

In particular, three of the authors of the present paper (BJN) were able to characterize, for all families of classical finite or affine Coxeter groups (A_n, $\tilde{A}_n$, B_n, etc.), the series $W_n^{FC}(q)$, and in fact, the bivariate generating function

$$W(x, q) := \sum_n W_n^{FC}(q)x^n, \tag{1}$$

by systems of non-linear q-equations [5]. For instance, in the A-case, the system can be reduced to a single quadratic q-equation for a series denoted $M^*(x) \equiv M^*(x, q)$:

$$M^*(x) = 1 + xM^*(x) + xq(M^*(x) - 1)M^*(xq).$$

Using these systems of equations, BJN were able to prove that $W_n^{FC}(q)$ is always a rational function of q, with simple poles at roots of unity: this means that the coefficients of these series are ultimately periodic [5,25]. This was first proved in the $\tilde{A}$-case by Hanusa and Jones [23]. Subsequently, one of us (N) showed that $W^{FC}(q)$ is in fact rational for any Coxeter group W, and determined when its coefficients are ultimately periodic [27].

The aim of this paper is to provide closed form expressions for generating functions of the form (1) for all classical Coxeter groups. Some of them turn out to be particularly elegant. Here are, for instance, our results in the A- and $\tilde{A}$-cases. For $n \geq 0$, we denote

$$(x)_n \equiv (x; q)_n = (1 - x)(1 - xq) \cdots (1 - xq^{n-1}). \tag{2}$$